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Two essays on the optimal control of infectious diseases: Examining discrepancies between discrete-time and continuous-time models

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Contents

Abstract

A growing body of literature on the optimal allocation of resources in controlling the spread of communicable diseases has garnered considerable attention during the last four decades. Although such literature is relatively unanimous formally speaking - *i.e.*, marrying tools of optimal control theory with epidemiological nonlinear models - it has been quite polarized over both the theoretical question of the choice of time scale (*i.e.*, discrete versus continuous) and, relatedly, the question of the most adequate optimization tool (Pontryagin's Maximum Principle versus Bellman's Dynamic Programming) to be employed in determining the lowest-cost policy for containing and eradicating the infection. This thesis theoretically investigates the roots of the discrepancies that exist between these two divided bodies of literature, seeking for ways to reconcile the results that are obtained by these two different approaches. The central analysis focuses on two pairs of articles on the control of SIS infections: i) two classical articles written in the 1970s that disagree on the pulsing behavior of the optimal policy over discrete and continuous time, and ii) two recent articles that examine the optimal allocation of funds between multiple connected populations when the social planner faces tight budgets, pointing out the difficulties that arise in analytically

solving the problem in continuous time. The implications of this theoretical investigation extend to similar models in topics as diverse as fishery management, corruption control, and crime prevention, while its practical contribution lies in carefully prescribing optimal intervention strategies for public health policymakers.

Keywords: optimal control - SIS model - time scale - Pontryagin's Maximum Principle - dynamic programming - strong Legendre-Clebsch necessary condition - Green's theorem - convergence of solutions - tight budgets - public health.

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CHAPTER 1

Introduction: Cure, no cure, half-cure?

"Where shall I begin, please your Majesty?" he asked.

"Begin at the beginning," the King said gravely...

Alice's Adventures in Wonderland, Lewis Carroll

In the increasingly interconnected world in which we live, infectious diseases represent a growing threat to global public health. Not only do such diseases remain a significant burden for most of the developing countries, but also their emerging drug-resistant varieties present a serious source of anxiety for the developed world. Lopez et al. (2006) have reported that five infectious diseases (HIV, malaria, tuberculosis, lower respiratory infections, and diarrheal diseases) were among the top ten global causes of death in 2001(Laxminarayan and Malani, 189). WHO vividly describes the magnitude of the global impact of infectious diseases in these terms: "Over the next hour alone, 1,500 people will die from an infectious disease over half of them being children under five."[1] Furthermore, in 2008, Trust for Americas Health published a report that announced a quite dramatic message: At least 170,000 Americans die each year because of newly emerging

[1] http://www.who.int/infectious-disease-report/pages/textonly.html

and reemerging infectious diseases, and this number is predicted to increase significantly in the coming decades due to the development of new drug-resistant infections, globalization, and the ongoing changes in the climate.[2] The need to understand optimal ways to control the spread of these diseases, therefore, presents itself stronger than ever in the twenty-first century.

1. Features of the economic approach to infectious diseases

The dynamics of infectious diseases[3] and the public health policies aiming to control these infections represent a very fruitful terrain that has only recently been explored by economists. Although epidemiology has provided well-equipped mathematical models of the dynamics of such infectious diseases for a long time, only recently have economists taken an interest in exploring optimal interventions to control and eradicate these diseases. The issue of controlling the spread of such diseases, nevertheless, represents one of the major challenges that public health policy makers face not only in the developing world, but in the developed world as well. Hence, a deeper formal analysis of the social costs of these infections and of the efficiency of various interventions is of primary importance in order to have a better

[2] "Germs Go Global: Why Emerging Infectious Diseases Are a Threat to America", Trust for Americas Health. http://healthyamericans.org/report/56/germs-go-global

[3] Throughout this thesis, the terms infectious disease, "communicable diseases", and infection are used interchangeably, although there exist slight biological differences between the three.

understanding of these infections and design more effective public health policies.

A coherent economic modeling of infectious diseases requires a careful evaluation of the multiple types of costs that these diseases impose upon the populations that they invade. The loss of lives is the single most important of these costs. While the lost lives cannot be easily assigned an equivalent monetary value that needs to be added to the other types of costs in order to calculate an approximate aggregate cost of the disease, often the number of lost lives is in itself a good approximation of this aggregate cost and can be used by itself as an objective function that needs to be minimized under certain constraints. Secondly, the reduction in the population and the inability of the infected individuals to proceed with the regular life activities for as long as they are infected are translated into forgone output, decreased labor productivity, and reduced consumption. Also, several scholars (Bleakley (2010), Miguel and Kremer (2004), Bobonis et al. (2006) etc.) who have contributed to the literature on the economic impact of infectious diseases have suggested that health has an indirect effect on the investment decisions regarding human capital, and hence income. A third type of impact includes costs related to the prevention and treatment methods employed to control the spread of the diseases. The particular (mathematical) form of these costs varies according to the specific characteristics of the disease and the control tools available for that disease. Lastly, because of the communicable nature of infectious diseases, infected individuals impose a social

cost upon the healthy individuals, that is, individuals who have not been infected yet, but who are susceptible to the disease due to their being surrounded by potential infecting agents. This externality effect that marks the spread of infections plays a crucial role in the dynamics that govern the spread of the disease and, consequently, in the policies employed to control such dynamics.

The particular characteristics of an infectious disease should be kept in mind when attempting to model the economic dynamics and the impact of such a disease. Infectious diseases are marked by a variety of characteristics, which makes a general formulation hard to establish. In terms of its evolution, an infection proceeds either to recovery and further susceptibility, immunity, or death. In terms of rates and modes of transmission, an infection might be transmitted at various rates mostly via direct contact between individuals or via vectors. The typical intervention methods used to control an infection are prevention and therapy. The most effective prevention method is vaccination, while some secondary less reliable prevention methods include the strict application of hygiene rules, avoidance of contact with the source of the infection etc. Therapy, on the other hand, consists of specific anti-infection medicines, which might be designed particularly for either early-stage or later-stage treatment of the disease. Several fatal diseases, such as HIV/AIDS, can be avoided only by taking preventive measures (so far), while for several other infections, such as cholera, malaria, and gonorrhea, treatment of the infected people is the only viable form of controlling

the disease. A mix of the two intervention forms is possible for most of other infections; this fact brings up questions about the optimal combination of prevention and therapy, and the circumstances under which one of these forms is superior to the other. An example of such infectious diseases is tuberculosis: the widely-used vaccine for TB is the Bacillus Calmette-Gurin (BCG) vaccine, while anti-TB treatment consists mostly of antibiotics such as isoniazid and rifampicin.[4] In addition, as mentioned earlier, infectious diseases vary across their costs of intervention, as well as across the target of the eventual intervention: the entire population, the infected, the uninfected, or the contact channels between the two. Therefore, facing such a myriad of characteristics, the existing economic-epidemiological models have focused only on selected tractable features that are crucial in the evaluation of cost-minimizing policies.

While the economic literature on epidemiological control is growing rapidly, the unique contribution that economists bring to this area of inquiry remains unique in two directions. First and most importantly, economists bring together behavioral choice and epidemiological dynamics to illustrate how personal decision making affects the prevalence of an infection in a feedback fashion [5]. Secondly, economists possess the necessary tools to frame the problem of epidemiological control as one of "a social planner's optimization

[4]World Health Organization: http://www.who.int/mediacentre/factsheets/fs104/en/index.html

[5]By feedback fashion, I mean that personal choices are both informed by and affect the prevalence level.

problem" and to obtain results for the optimal interventions. The insights that economic analysis is providing to the traditional epidemiological models are changing fundamentally the way both economists and epidemiologists think about infection dynamics and infection control. While economic epidemiology has already made some substantial advances, further research on this field is of great significance and interest.

The most widely used epidemiological compartmental model[6] in the economic literature is the SIS (susceptible-infected-susceptible) model. The SIS model assumes that individuals do not become immune to the infection after recovery; instead, they join the pool of susceptible individuals again and can be reinfected through future contacts with infected individuals. This model is particularly useful in describing the evolvement of diseases such as gonorrhea, meningitis, plague, streptococcal sore throat, malaria, and sleeping sickness. (Anderson et al. 2011, 1) In the simplest version of the model with a fixed population, the entire population is comprised of infected and susceptible individuals only, and individuals can move back and forth between the states of being infected and being susceptible.[7] Therefore, changes in the number of infected individuals are driven by three main factors: 1) the rate

[6]The two compartments in the SIS model are: "Susceptible" and "Infected".

[7]The model was first introduced by Kermack and McKendrick, in their well-known article: Kermack, W. O. and A. G. McKendrick, (1927) "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society of London Series A*, 115, 700-721.

of spontaneous recovery of infected individuals (i.e. recovery without treatment), 2) the infectiousness of the disease (the rate of susceptible-infected contacts that cause the infection of the susceptible individual), and 3) the fraction of the infected subpopulation that gets treated and the efficiency of the treatment. On the other hand, a positive change in the infected subpopulation implies an equal negative change in the susceptible subpopulation, so a description of the dynamics of one of the subpopulations is sufficient (in the simple case of fixed population). A rigorous mathematical presentation of the SIS dynamics is presented in the next section of this chapter.

2. Advances in the literature

The growing economic literature on infectious diseases can be classified into four main strands of research (Laxminarayan and Malani, 2011). The first strand of the literature is the examination of the direct and indirect impact of infectious diseases on income and economic development. A second strand of research deals with the role that individual incentives play in the dynamics and control of infectious diseases; the questions asked in this neighborhood of the literature revolve mostly around three main topics: 1) the correlation between the prevalence level of the disease and the self-protective (analogously, the risk-taking) behavior displayed by the individuals that are exposed to the disease, 2) the demand for treatment and vaccination, in light of the positive externalities that they bring to the entire population, and 3) the demand for information about one's state of the

disease, i.e. the demand for testing, which can be either voluntary or mandatory. The third strand of the literature focuses on institutional and national incentives for controlling infectious diseases, analyzing the particular contextual framework in which institutions and nations respond to outbreaks of infections and the effect that the incentives arising out of such contextual frameworks have on the optimal level of the control effort.

The fourth strand of research that has garnered considerable attention in the literature on economics of infections is the examination of optimal allocation of resources to control and fight communicable diseases. The first economic articles on this topic appeared in the early 70s, when optimal control theory and dynamic programming tools had just started to establish their influence in the context of epidemiological models. While the work that has been done on this topic can be classified differently using different criteria, two of those criteria have substantially shaped the debate in this literature: 1) the number of the populations over which the treatment effort is optimized, and 2) the flow of time in these optimization problems (i.e. continuous-time vs. discrete-time models). This thesis finds its inspiration in a long-standing controversy around these two criteria.

The main focus of this thesis is centered around two classical articles published in the early 1970s in *Biometrics* by Sanders (1971) and Sethi (1974), which aim to evaluate the socially optimal level of treatment in a single population facing linear costs of the disease–dependent on the number of infected people and the chosen treatment effort level–and inequality

constraints in the treatment level. The two analogous optimization problems formulated in discrete time and continuous time respectively are the following:[8]

$$\begin{aligned} &\underset{\{\gamma_t\}}{\text{minimize}} && \sum_{t=0}^{T} \alpha^t (Cx_t + K\gamma_t) \\ &\text{subject to} && x_{t+1} = x_t + \beta x_t (N - x_t) - \gamma x_t, \\ &\text{with} && 0 \leq x_t \leq N,\ 0 \leq \gamma_t \leq b, \\ &\text{and} && x_0 \text{ given.} \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\underset{\{\gamma_t\}}{\text{minimize}} && \int_0^T e^{-\alpha t} (Cx + K\gamma) \\ &\text{subject to} && \frac{dx}{dt} = \beta x (N - x) - \gamma x, \\ &\text{with} && 0 \leq x_t \leq N,\ 0 \leq \gamma_t \leq b, \\ &\text{and} && x_0 \text{ given.} \end{aligned} \tag{1.2}$$

Notice that in the problems above, x denotes the number of infected individuals in the population, β denotes the infectiousness rate of the disease, N denotes the total number of the population, γ reflects the level of the treatment effort chosen to be applied in the population, C and K represent

[8]The variable α is the intertemporal discount factor in discrete time. If we let $\alpha = \frac{1}{1+r}$ in discrete time, the corresponding continuous-time discount factor will be of the form e^{-r}. Notice the abuse of notation in the use of the symbol α in the discrete-time formulation and the later continuous-time formulation.

the marginal costs of an additional infected individual and an additional unit of treatment level respectively, and b denotes an upper bound of the possible treatment level that can be attained by the society due to its technological limits. Referring to the form in which γ is incorporated in the model, Sanders states that "the impact [of the level of program effort] is proportional to the number infected at that time" (Sanders, 885). Finally, this formulation of the differential/difference equation of motion deviates from the classical Kermack-McKendick formulation in that it does not account for spontaneous recoveries, hence it implicitly assumes that everyone that is infected will return to his susceptible state due only to effective treatment.

The article by Sanders (1971) is one of the earliest articles on the application of mathematical modeling techniques to the class of control problems that are concerned with a health delivery system geared toward the elimination of a particular health problem(Sanders 1971, 883). Sanders introduced the linear cost function of the form $(Cx + K\gamma)$, where the first term captures the overall costs that the society bears due to the x infected individuals present in it, while the second term represents how costly a chosen level of program effort is. This early article posed the problem in discrete time, and employed a dynamic programming argument to solve for the optimal treatment level. The main line of the argument is based on the crucial fact that although the single-period costs are linear in both the infection level (the state variable) and the treatment level (the control variable), the value function is strictly concave with respect to the infection level due to the

concave SIS equation of motion, which enters the argument because of the forward-looking nature of dynamic programming. Concavity ensures that the optimal policy will be of a bang-bang form, i.e. the treatment level in any period within the considered finite horizon will be either at the maximal level or at the minimal level, but never at an intermediate one. A social planner should either choose to apply the maximum treatment effort that the society is capable of, or he should not intervene at all in the proliferation of the disease.

Sethi responded to the discrete-time solution offered by Sanders by transferring the problem in continuous time in the form of Equation (1.2). In justifying his decision to transfer the problem to another time framework, he argued that dynamic programming,–the continuous form of which was highly unexplored at the time–was the problematic optimization tool employed by Sanders that led to faulty results. On the other hand, Pontryagin's maximum principle, originally introduced in its continuous form (and whose discrete version was formulated rigorously only several years later), was the preferred continuous-time solution method at the time. He employed Miele's interpretation of Green's theorem in extremization of line integrals in order to solve for an optimal intermediate level of treatment in the case when the upper bound b is large enough. He then returned to the method of switching functions–widely explored and used in the late 1960s as a tool of identifying singular solutions when the Hamiltonian is linear in the control,–to prove the optimality of the singular solution in a more general setting.

While there is no obvious reason why the method of dynamic programming fails to produce the correct optimal control path in discrete time, the singular control path that Sethi identifies as optimal in continuous time openly disappoints our pre-conceived intuition that optimal solutions in discrete time and continuous time should be of similar form, even if obtained via different optimization methods. Taking for granted that both methods have been applied correctly – an assumption which will be carefully scrutinized in the next chapter – this discrepancy between the solutions in discrete and continuous time might possibly result from the special structure of this particular optimization problem. Anderson and Salant (2011) consider the discrete-time problem introduced above and check the optimality of the singular (turnpike) solution identified by Sethi in discrete time. They do so by supposing that when the level of the infection is at the turnpike level, a small perturbation h from the singular policy should increase the costs of the program as the singular control is supposedly the optimal policy. They conclude that in this discrete-time framework, Sethi has mistaken a local maximum for a global minimum, because any perturbation around $h = 0$ produces lower costs than the case of $h = 0$. This is a very important result towards the goal of understanding the source of the differences between the policies suggested by Sethi and Sanders, but notice that this analysis is strictly limited to the discrete-time framework. In other words, Anderson and Salant have shown that the intermediate level solution proposed by Sethi is not optimal in the discrete-time formulation of Sethi's problem, but

this does not imply much about the continuous-time framework in which he originally operates.

This debate has been enriched further only recently with the emergence of a new gap in the literature, seemingly related to the first one. Considering a similar optimization problem of linear cost and SIS dynamics, Rowthorn et al. (2009) and Anderson et al. (2011) analyze the case of the optimal allocation of a tight budget between two subpopulations of a fixed population living in two interconnected regions in continuous and discrete time, respectively. Rowthorn et al. conclude via numerical methods that the optimal path to pursue is to allocate treatment to the subpopulation with the lower level of infection. They point out a major technical difficulty in formulating analytical solutions for optimal control problems that involve the SIS model: because the SIS model of disease dynamics contains a nonconvexity, the standard use of sufficiency conditions to find an analytical solution fails. Therefore, they turn to numerical approximation methods in order to obtain an optimal solution for this problem. While unable to pin down analytical optimal solutions for their model, the authors still manage to analytically prove that the worst possible path is to adjust treatment levels that equalize the level of infection in each subpopulation. Anderson et al. address this technical difficulty by introducing the assumption of tight budgets [9] and

[9]This discussion will consider the case of tight budgets only when the transfer of funds across time in the form of borrowing or lending is not possible. The relaxation of this assumption and its implications present an interesting path for further research.

employing a modified dynamic programming approach to obtain analytical solutions for the discrete-time social planner's problem. The article models the budget-allocation decisions that a health authority makes in controlling and combating an SIS-governed infectious disease that spreads in a finite number of unconnected and interconnected populations (analogous to the subpopulations in the model of Rowthorn et al). The health authority has to decide how many infected people from each population should be treated in every time period. The case of a tight budget granted every period of time and the case of a certain amount of wealth granted only at the beginning are discussed separately in the paper. The aim of the health authority is to minimize the discounted social cost of the infection in all the populations, which is expressed as a linear combination of the respective infection levels. The authors conclude that it is optimal, under a tight budget, to focus entirely on one group at a time, rather than treating infected individuals from several groups simultaneously. In this sense, the optimal solution that they propose is of a bang-bang form: interior solutions–that is, combinations of treatment effort in two or more subpopulations in the same time period–are never optimal. Furthermore, they argue that the groups that should have priority in receiving treatment are those with the lowest levels of infection.

3. Questions and structure of the thesis

The main goal of this thesis is to synthesize a unified understanding of the optimal policies that need to be followed in controlling infectious diseases, focusing on the infections whose dynamics are governed by the SIS epidemiological model. The discussion develops separately along two main strands: 1) the attempt to bridge the gap between the discrete-time and the continuous-time frameworks offered by Sanders and Sethi, by examining possible ways of resolving their disagreement on whether the optimal treatment level follows a bang-bang path or a singular path, and 2) the implications that the earlier debate between Sethi and Sanders has on the extension of the modified discrete-time dynamic programming approach used by Anderson et al. to continuous time in order to overcome the difficulties identified by Rowthorn et al. (or alternatively, the exploration of potential virtues of the discrete-model that might allow the approach to be successful in discrete time but not in continuous time).

The major questions that will guide the discussion throughout this work can be categorized in three general families of questions:

- What are the particular features of this optimization problem, how are these features affected by the chosen time scale, and how do these features contribute to potential incongruence between optimal solutions in discrete and continuous time? From a chronological standpoint, have the methods of calculus of variations, optimal

control theory, and dynamic programming been able to cope with this particular type of problem successfully in the past, and if so, how?

- Is the disagreement between Sethi and Sanders a purely methodological one–that is, is either Pontryagin's maximum principle or Bellman's dynamic programming particularly ill-suited to this type of problem due to their classical underlying assumptions,–or is their disagreement rooted more deeply into a potential time-step-dependent nature of the optimal solution? How do continuous dynamic programming and the discrete maximum principle handle the problem? In the spirit of the discrete-time variations around the steady-state solution used by Anderson and Salant, can we find a continuous-time numerical example of an extremal policy that dominates the steady-state solution offered by Sethi? How do the strategies of chattering and pulsing, which are more similar to a continuous-time version of the optimal solution offered by Sanders, perform compared to the steady-state solution identified by Sethi?
- How is the problem formulated by Sethi and Sanders similar and/or different from the problem formulated by Rowthorn et al. and Anderson et al.? Can the solution method offered by Sethi in continuous time help with the problem posed by Rowthorn et al.? Do we expect similar divergences between optimal solutions in different time frameworks for this problem as well? How can the modified

dynamic programming approach offered by Anderson et al. for the discrete-time problem be extended in continuous time?

This thesis is organized in five chapters; the introductory chapter you are reading now is the first one. Chapter 2 offers a technical discussion of the major mathematical concepts, techniques, and proofs useful for the rest of this work. Chapter 3, which is the most substantial chapter of this thesis, offers an analysis of the disagreement between Sanders and Sethi, providing some attempts to identify potential sources of this disagreement and connect the work of the two. Chapter 4 turns to the more recent pairs of papers by Rowthorn et al. and Anderson et al., attempting to apply lessons drawn from Chapter 3 to the discrepancy between solutions in discrete and continuous time identified in these papers. The fifth chapter concludes this work with a summary of useful technical results, policy recommendations, and persistent old and new questions.

4. Further comments on the relevance of this work

The theoretical assumptions made in the articles that will be discussed extensively in this thesis and the questions that are built upon those assumptions are not of academic interest only. They are motivated by actual economic aspects of infectious diseases. In this section, we provide some facts to illustrate this point.

Sanders provides an example of an infectious disease that illustrates his assumption about the cost function that he chooses to use in his model.(Sanders

1971, 887) Trachoma, a disease that is highly prevalent among the Papago Indian people of Southern Arizona, is a disease that affects the cornea of the eye and if not treated on time, can degrade to complete vision loss. Considering data from a trachoma control program that was applied in school age children in the San Xavier Reservation during the period 1964-1967, he interprets the variable γ in his model as the screening effort needed to identify the infected individuals in the population, and $K\gamma$ as the cost of a total screen. Furthermore, he argues that other assumptions of the model are very realistic in this setting as well, such as the fact that: a)there is no immune state for trachoma, i.e. people that have been treated once are still susceptible to it, b)the population is naturally isolated, c)the infected and the susceptible subpopulations are not isolated from each-other, d)the program predicted a regular screening procedure every 6 months, and e)the treatment used in treating people diagnosed with the disease is effective only in about 60 percent of the cases.

The literature on the optimal treatment level in a single population is particularly relevant to endemic infectious diseases that prevail in isolated populations. The case of trachoma is one illustration of such endemic infections. Furthermore, the issue of controlling infections in one population is fundamental for decision making at the level of national health ministries. On the other hand, the issue of treating multiple connected (sub)populations simultaneously is of great interest in the context of globalization and the

need to control communicable diseases across boundaries. International policy making institutions, such as WHO, are continually interested on how to divide funds across several countries or regions with different prevalence levels of the same disease.

Also, the assumption of tight budgets that Anderson et al. make in their article (which then allows them to establish an analytical solution for the problem) is consistent with the limited treatment resources that countries and organizations possess. It is quite unrealistic to suppose that we can treat every infected individual at the same time. As Laxminarayan and Malani (199) point out, there is not enough funds to treat everybody that has been infected by a disease, even for a disease such as HIV which has absorbed enormous funds for treatment over years. Another illustration of this point comes from Zambia. Zambia has one of the best-funded malaria programs in Sub-Saharan Africa, yet the percentage of the children who receive effective treatment does is not higher than 13 percent.

To conclude, the relevance of this work lies in three main directions. First, the reconciliation of the discrete-time and continuous-time optimal solutions for the type of optimization problems with linear objective function and nonlinear dynamics is of theoretical interest. This type of problems is widely used in contexts where population dynamics need to be modeled. Secondly, the results of this thesis might be particularly useful for public health practitioners, whose work constantly aims to control diseases that spread in continuous time through decisions made in discrete time. The

examples described above argue further about the relevance of this problem for the public health field. Thirdly, similar types of problems arise in other subfield of economics as well, such as resource economics, economics of corruption etc. A better understanding of the problem at hand will possibly shed light on solutions to these other problems as well.

CHAPTER 2

Mathematical preliminaries

"Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be."

Bertrand Russell

"I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives."

Charles Hermite, in a letter to Thomas de Stieltjes

In this chapter, several techniques and results that are important mathematical prerequisites to the analysis developed in the next two chapters will be reviewed. The chapter serves two purposes: 1) the general introduction of basic ideas in optimization, and 2) the exposition of some particular mathematical results that are directly related to our problem of interest. Section 1 and 2 offer a brief discussion of dynamic programming, the Maximum Principle, and the connection between the two. Section 3 explains the basic technique of extremization of line integrals by means of Green's theorem. Section 4 reviews the particular features of singular control problems, following the classic work of Bell and Jacobson (Bell and Jacobson, 1975). Section 5, 6, 7, and 8 discuss three other topics that are crucial tools in

understanding the work of Sanders and Sethi discussed in the next chapter: the current-value Hamiltonian function, Euler's method of approximation, the logistic nature of the SIS dynamic equation, and the discretization of the discounting factor. Readers who are familiar with the mathematical preliminaries may omit this chapter and go directly to the next chapter.

1. Divide and conquer: Dynamic programming

Dynamic programming – an optimization technique developed in the mid-1950s by Richard Bellman (Bellman, 1957) – exploits the ability of a multi-stage decision problem to be broken down in smaller optimization subproblems that can be solved separately with greater ease. A dynamic programming problem is defined in terms of *states*, *decisions (alternatively, actions or controls)*, and *momentary rewards or costs*. Once the tree of states and decisions, along with accompanying rewards for each state-decision pair, is known, we can define a sequence of *value functions* $\{V_i(x)\}$, which represents the optimal value of a state x at a certain moment in time $i \in \{0, 1, ..., n\}$, given that the decision maker will act optimally from moment i and on. This concept of a value function allows for a recursive definition of the decision-making process, as $V_{i-1}(x)$ is equal to the sum of the one-period reward (or cost) that the agent obtains from acting optimally (making decision u^o [1]) from time $i-1$ to time i, and the subsequent $V_i(x')$, where x' is the state that results from acting u^o from state x. This

[1]The superscript o stands for "optimal".

recursive relationship is captured by Bellman's Principle of Optimality, the mathematical form of which will be presented in the next subsection.

1.1. The principle of optimality. Consider a deterministic system governed by the following difference equation:

$$x_{i+1} = x_i + f_i(x_i, u_i) \qquad i = 0, 1, ..., N-1$$

$$x_0 = \bar{x_0}$$

Notice that as N becomes infinitely large, the optimizing horizon becomes infinite. For convenience, we consider only the finite horizon dynamic programming problem. In the general case, the state variable x_i is an n-dimensional vector, the control variable u_i is an m-dimensional vector, and f_i is an n-dimensional function of the pair (x_i, u_i). In order to simplify our discussion, we will consider the case when x and u take only scalar values, and f_i is a two-dimensional function of the same form independent of i, so $f_i = f$ for all i. Lastly, our discussion considers only the case of costs that need to be minimized; an analogous discussion that involves rewards would follow from this.

A *control trajectory* (or control path) is a sequence of control actions over time $\{u_i, i = 0, 1, ..., N-1\}$. Similarly, a *state trajectory* (or state path) is a sequence of states of the form $\{x_i\}$. All the admissible control actions form a control space $\mathcal{C}$. Similarly, all the attainable states live in the state space $\mathcal{S}$.

The *cost function* (or the performance index) of a system with initial state x_0 and control path $\{u_i\}$ is defined as:

$$V_0(x_0, \{u_i\}) = \sum_{i=0}^{N-1} \alpha^i C_i(x_i, u_i) + \alpha^N C_N(x_N)$$

where C_i represent the one-period costs, C_N represents the cost of ending up in the last state x_N, and α is the discounting factor of the future costs.

The major goal of the problem is to find a control path $\{u_i\}$ that minimizes $V_0(\bar{x_0}, \{u_i\})$ for any given initial state. Let $\{u_i^o\}$ be the optimal control path starting from the initial state $\bar{x_0}$, which results in the corresponding state trajectory $\{x_i^o\}$. Let $\pi^o = \{g_i^o\}$ be the optimal policy, which is a sequence of the optimal control laws $u_i = g_i(x_i)$ for every period i. At this time, we can define the optimal value function $V_i^o(x_i)$ to be the total future cost of being in state x_i at time i, given that the optimal policy will be followed from period i to the last period N. Similarly, $V_0^o(x_0)$ is the minimized total cost of starting from an initial state x_0.

The principle of optimality states the recursive nature of the value function:

$$V_i^o(x_i) = \min_{u_i}[C_i(x_i, u_i) + \alpha V_{i+1}^o(x_i + f(x_i, u_i))] \qquad \text{where} \qquad V_N^o(x_N) = C_N(x_N)$$

This forward-looking minimization procedure will yield the optimal control path for the remaining periods from i to N, given that the system is at state x_i at time i. So, we can obtain the optimal control path for a given x_0. If this procedure is repeated over all the possible initial states $x_0 \in \mathcal{S}$,

we will obtain the functions $V_i^o(\cdot)$ and $\pi(\cdot)$ by using an iteration procedure for either the value function or the policy function.

1.2. Continuous dynamic programming and the Hamilton-Jacobi-Bellman equation. In its beginnings, the dynamic programming principle was formulated in discrete-time. Further work carried this principle to continuous time, where the Hamilton-Jacobi-Bellman equation is the analogous continuous-time formulation of the discrete-time Bellman equation. The Hamilton-Jacobi-Bellman (HJB) equation, which can be considered as an extension and a combination of the work done in classical physics on the Hamilton-Jacobi equation and Bellman's work on discrete-time dynamic programming, is a partial differential equation whose solution is the value function introduced in the last subsection.

Consider the following continuous-time cost function:

$$V(t=0, x_0, \{u_t\}) = \int_0^T e^{-\alpha t} C(x_t, u_t) dt + e^{-\alpha T} D(x_T)$$

We need to minimize $V(t, x_0)$, subject to the differential equation

$$\frac{dx}{dt} = f(x_t, u_t).$$

Notice that this differential equation is analogous to the difference equation from the last subsection. Suppose that u^* is the optimal control path. Then, $V(t, x) = V(t, x, \{u^*\})$ is the continuous-time counterpart of $V_i^o(x)$, and it is also the solution of the HJB equation. Assuming that the value function is everywhere continuous and differentiable with respect to t and x, the HJB

equation can be written in the following form:

$$\min_{u\in\mathcal{U}}\{C(x,u)+\frac{\delta V(t,x)}{\delta t}+\frac{\delta V(t,x)}{\delta x}f(x,u)\}$$

In order to understand the derivation of the HJB equation from the discrete-time Bellman equation, the following argument is provided. The argument does not prove all the technical details of this derivation; it just sketches the reasoning of the derivation. Suppose that the horizon $[0,T]$ is partitioned in N parts of length δ each. Then, $V(T,x)$ is approximated by $\tilde{V}(N\delta,x)$. Then, $\tilde{V}(k\delta,x)=\min_u\{\delta C(x,u)+\tilde{V}((k+1)\delta,x+\delta f(x,u))\}$. The Taylor series expansion for a certain function $g(x,y)$ is:

$$g(x+\Delta x,y+\Delta y)=\sum_{i=0}^{\infty}\{\frac{1}{i!}[\Delta x\frac{dg}{dx}+\Delta y\frac{dg}{dy}]^i g(x,y)\}$$

Therefore,

$$\tilde{V}((k+1)\delta,x+\delta f(x,u))=\tilde{V}((k\delta+\delta,x+\delta f(x,u))=\sum_{i=0}^{\infty}\{\frac{1}{i!}[\delta\frac{d\tilde{V}}{dt}+\delta f(x,u)\frac{d\tilde{V}}{dx}]^i\tilde{V}(k\delta,x)\}$$

This infinite sum can be written as:

$$\tilde{V}((k\delta+\delta,x+\delta f(x,u))=\tilde{V}(k\delta,x)+\delta\frac{d\tilde{V}(k\delta,x)}{dt}+\delta f(x,u)\frac{d\tilde{V}(k\delta,x)}{dx}+O(\delta)$$

But, we also know that:

$$\tilde{V}(k\delta,x)=\min_u\{\delta C(x,u)+\tilde{V}((k+1)\delta,x+\delta f(x,u))\}$$

Therefore,

$$\tilde{V}(k\delta,x)=\min_u\{\delta C(x,u)+\tilde{V}(k\delta,x)+\delta\frac{d\tilde{V}(k\delta,x)}{dt}+\delta f(x,u)\frac{d\tilde{V}(k\delta,x)}{dx}+O(\delta)\}$$

Substracting $\tilde{V}(k\delta, x)$ from both sides and dividing both sides by δ, we get:

$$0 = \min_u\{C(x,u) + \frac{d\tilde{V}(k\delta,x)}{dt} + f(x,u)\frac{d\tilde{V}(k\delta,x)}{dx} + O(\delta)\}$$

As $\delta \to 0$ and $k \to \infty$ such that $k\delta = T$, assume that

$$\lim_{\substack{\delta\to 0\\ k\to\infty\\ k\delta=T}} \tilde{V}(k\delta, x) = V(t,x).$$

From this assumption and the previous result, we obtain the HJB equation:

$$0 = \min_u\{C(x,u) + \frac{V(t,x)}{dt} + f(x,u)\frac{V(t,x)}{dx} + O(\delta)\}$$

It is generally very hard to obtain solutions for this partial differential equation. Nevertheless, the HJB equations is a necessary and sufficient condition for optimality, therefore in case a solution is found, it is guaranteed that that solution for the value function will yield the minimizing policy. Secondly, solutions of the HJB equation are usually nonsmooth functions, i.e. value functions that are not differentiable everywhere. The previous proof was constructed upon the assumption that the value function is differentiable everywhere, therefore a lot of work has been done in order to circumvent this difficulty. The concept of viscosity solutions to the HJB equation has been introduced as a remedy. A dicussion of viscosity solutions is beyond the scope of this chapter, but a complete treatment of this topic can be found in Bardi and Capuzzo-Dolcetta (1997).

2. Concepts in optimal control theory

Optimal control theory arose out of the inability of the calculus of variations, the classical method in dynamic optimization, to deal with corner solutions, not-everywhere-differentiable state trajectories, and constraints in the control variable. The admissible control trajectories in optimal control theory are required to be only piecewise continuous, not necessarily everywhere continuous. The corresponding state trajectory of a piecewise continuous control trajectory will be differentiable everywhere except for the points where the control trajectory is discontinuous. Therefore, optimal control theory is well-equipped to solve problems with discontinuous optimal control paths. A second convenient feature of optimal control theory – as Chiang (2005, 161-164) notes in his introduction of optimal control theory – is its ability to handle optimization problems with control constraints, which might, for instance, be in the form of a bounded and closed control space (such as an interval in the case of a scalar control variable). These two features of optimal control theory are very important to our analysis because they allow this method to identify bang-bang solutions: piecewise continuous solutions where each of the continuous pieces takes a value from the boundary of the control space.

The basic problem of optimal control theory can be written in the following form:

$$
\begin{aligned}
&\max_{\{\gamma_t\}} && \int_0^T U(t,x,u)dt \\
&\text{subject to} && \frac{dx}{dt} = f(x,t,u), \\
&\text{with} && x(0) = a,\ x(T) \text{ free};\ a, T \text{ given} \\
&\text{and} && u \in \mathcal{U} \text{ for all } t \in [0,T].
\end{aligned} \tag{2.1}
$$

In the above problem, U and f are continuous in all their arguments and have first-order partial derivatives with respect to x and t, but not necessarily with respect to u. (Chiang, 165) Also, the problem is written in the form of a maximization problem, but it can be easily transformed into a minimization problem by considering the negative of the momentary utility $-U(t,x,u)$ as the integrand in the first line of the problem. In this way, the problem is transformed from a problem of maximization of rewards into a problem of minimization of costs.

2.1. The Maximum Principle: Basic notions. The most useful result of optimal control theory is Pontryagin's Maximum Principle, developed in the early 1960s independently by both L. S. Pontryagin and his collaborators in former USSR[2], and M. Hestenes in the United States[3] . In order

[2]Boltyanskii V.G., R.V. Gamkrelidze. L.S. Pontryagin: "Towards a theory of optimal processes", (Russian), *Reports Acad. Sci. USSR Vol. 110(1)*, 1956

[3]Hestenes, M. R., "A General Problem in the Calculus of Variations with Applications to Paths of Least Time," *Rand Corporation RM-100, ASTIA Document No. AD 112382*, Santa Monica, California: 1950.

to introduce the Maximum Principle, the concepts of Hamiltonian function and costate variable need to be explained first. After presenting the mathematical form of the Maximum Principle, an economic digression follows, which provides the reader with a brief chart of the economic significance of this principle.

Keeping the same notation throughout this section, the Hamiltonian function is defined as:

$$H(t, x, u, \lambda) = U(t, x, u) + \lambda(t) f(x, t, u).$$

The costate variable in the system, λ, can be viewed as the continuous-time analogue of a Langrange multiplier, and it is a function of time t. So, we can talk about a costate path: the evolution of the costate variable over time. The conditions of the maximum principle are the following:

$$\begin{aligned} &\max_u H(t, x, u, \lambda) \text{ for all } t \in [0, T] \\ &x' = \frac{\delta H}{\delta \lambda}, \\ &\lambda' = -\frac{\delta H}{\delta x} \\ &\lambda(T) = 0. \end{aligned} \tag{2.2}$$

The second line is the equation of motion for the state variable x (which is $f(x, t, u)$), expressed as a partial derivative of the Hamiltonian with respect to the costate variable. The third line is an equation of motion for the costate variable. The fourth line represents the transversality condition:

the condition that the costate variable needs to meet at the end of the time horizon.

While we need to find u^* that maximizes the Hamiltonian function, that is not necessarily equivalent with the condition $\frac{\delta H}{\delta u} = 0$. That is because the Hamiltonian function is not necessarily everywhere differentiable with respect to u, and even when it is, the first-order condition $\frac{\delta H}{\delta u} = 0$ might identify a minimum instead of the desired maximum. In such a case, the maximum has to be searched at the boundaries of the control space. Another pathological case would be the case when the Hamiltonian is linear in u. If the control set is a closed set, then the optimal solution might be a corner solution.

Lastly, the following chart will provide a mapping of the concepts of the maximum principle into a classical economic example from capital theory. The chart is based on the discussion of Robert Dorfman in his-well known article "An Economic Intepretation of Optimal Control Theory".[4] In Dorfman's discussion, the basic problem is the decision problem of a firm that aims to maximize profits over a time horizon under the capital constraints it faces. In every moment, the firm has a capital stock that needs to be managed. The decisions of the firm (that can vary greatly, from decisions

[4]Dorfman, Robert. "An Economic Intepretation of Optimal Control Theory." *American Economic Review*. December 1969: 817-31.

concerning the rate and price of the output to decisions concerning the design of the product) affect the rate at which the size of this capital stock is changing.

Concept in Maximum Principle	**Economic Significance**
state variable x	amount of capital
control variable u	rate of change of capital
costate variable λ	shadow price of capital
$\lambda(0)$	shadow price of a unit of initial capital
$\lambda(T)$	shadow price of a unit of terminal capital
function U	current profit
$\lambda(t)f(x,u,t)$	future profit effect of policy u
Hamiltonian function H	overall profit prospect: (current profit) + (shadow price) $\cdot$ (change in capital corresponding to policy u)
$f(x,u,t) = x'$	rate of change of capital per unit of time due to the present amount of capital, policy u, and moment t
λ'	rate of decrease of shadow price per unit of time (depreciation of shadow price)

$x' = \frac{\delta H}{\delta \lambda}$	the way the policy decision affects the rate of change of capital: the change in capital is equal to the contribution of the shadow price to overall profits.
$\lambda' = -\frac{\delta H}{\delta x}$	shadow price depreciates at the rate at which capital contributes to overall profits.
$\lambda(T) = 0$	shadow price is driven down to zero at the end of the time interval, i.e. the left-over capital has no economic value to the firm.
Alternative transversality condition: $\lambda(T) > 0$ and $(x(T) - x_{\min})\lambda(T) = 0$	The shadow price is not driven down to zero (the firm intends to continue its existence beyond the optimizing horizon), but the terminal capital should be $x_{\min}$.

3. Green's Theorem and the extremization of line integrals

This discussion of the application of Green's Theorem to the extremization of line integrals will be based on the Pierre's treatment of this topic (Pierre 1986). Consider two real-valued functions of the form: $V_1 \equiv V_1(x, s, t)$ and $V_2 \equiv V_2(x, s, t)$. Let s be an explicit function of x and t. Then, we need to extremize the quantity:

$$\Delta = J_a + J_b = \int_a (V_1 \frac{dx}{dt} + V_2)dt + \int_b (V_1 \frac{dx}{dt} + V_2)dt$$

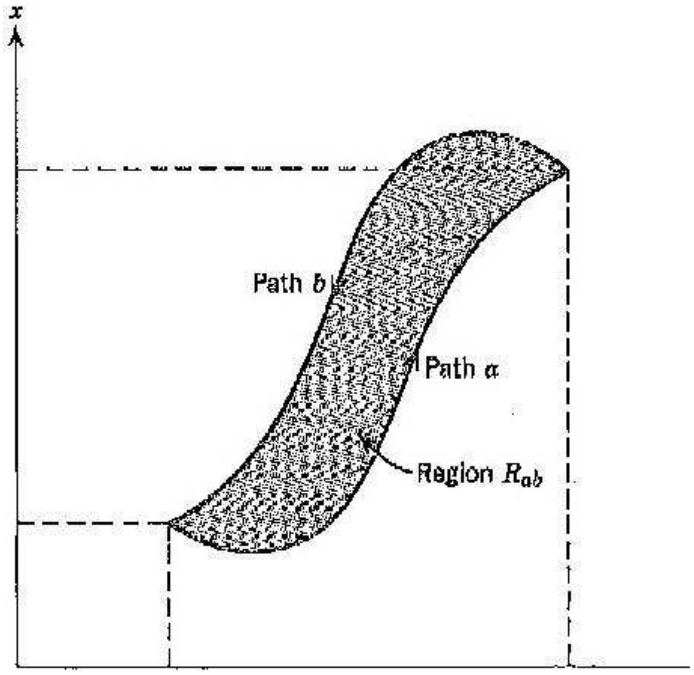

FIGURE 2.1. Paths and region of integration [Graph from Pierre 1969, 116]

Consider R_{ab} to be the region enclosed by the paths a and b, and also consider V_1, V_2, and s to be analytic functions on a and b. Then, the previous line integral can be written as the following surface integral:

$$\Delta = \iint_R (\frac{dV_2}{dx} + \frac{dV_2}{ds}\frac{ds}{dx} - \frac{dV_1}{dt} - \frac{dV_1}{ds}\frac{ds}{dt})dxdt$$

The paths over which we are integrating are showed in the following picture:

The sign of the double integral is dependent on the direction of the paths a and b; if a and b would have opposite directions from what is shown in the graph, then the double integral would be of negative sign. The sign of the integrand of the surface integral determines the value of x that minimizes/maximizes the initial line integral. If the integrand is positive above

the path a, negative below it, and zero along it, then J_a is the maximum value of J. On the contrary, if the integrand is negative above the path a, positive above it, and zero along it, then the path a is a minimizing path of J.

4. The curious case of singular optimal control problems

A singular minimizing control path for the general optimal control problem is defined as one for which the classical Legendre-Clebsch condition is not satisfied with strict inequality (Bell and Jacobson 1975). Therefore, along a singular control path, there is nothing we can say about the convexity of the Hamiltonian function with respect to the control variable. Goh (Goh 1966, Bell and Jacobson 1975) was the first to establish that the extremal trajectory is singular for the case in which the Hamiltonian H is linear in one or more elements of the control function $u(t)$.

DEFINITION. Consider u_i to be an optimal singular element of the control vector u on $[t_1, t_2]$ which appears linearly in H. Suppose that u_i appears explicitly in H after taking the time derivative of $\frac{\delta H}{\delta u_i}$ $2k$ times. Then, the integer k is called the *order of the singular control path*.

DEFINITION. Assuming that all the components of the control vector u are simultaneously singular, then u is called a *totally singular* control path if $\frac{\delta H(\bar{x},\lambda,t)}{\delta u} = 0$ for all $t \in [0, T]$. The control path is *partially singular* if

$\frac{\delta H(\bar{x},\lambda,t)}{\delta u} = 0$ holds for k intervals, whose sum of lengths is less than the total length of the optimizing horizon.

After having introduced these general concepts involved in singular optimal control problems, we turn to a brief discussion of singular solutions of autonomous (time-independent) optimization problems with nonlinear dynamic systems and linear utility function.

Suppose that the control variable u is a scalar. Let $x' = f(x) + g(x)u$ and $U(x, u) = ax + bu$. Then, the Hamiltonian function is:

$$H = ax + bu + \lambda(f(x) + g(x)u) = ax + \lambda f(x) + u(b + \lambda g(x)).$$

Taking the derivative of H with respect to u, we obtain:

$$\frac{\delta H}{\delta u} = b + \lambda g(x).$$

$W = b + \lambda g(x)$ is called a *switching function.* This switching function does not directly determine a stationary control, because it does not depend on u. Nevertheless, we can find $u(t)$ over a finite interval so that the switching function is zero over that interval. In order to do so, we need to take time derivatives of the switching function, and set them equal to zero.

The first time derivative of W does not depend on u. Given that the coefficient before u in the second time derivative will not be zero, we can determine an expression for the stationary solution u^*. In order to check the optimality of this singular control, we introduce the following necessary condition.

4.1. Generalized Legendre-Clebsch condition. A necessary condition for minimality is the usual convexity condition (second-order condition):

$$\frac{\delta}{\delta u}(\frac{\delta H}{\delta u}) \geq 0 \qquad \text{(Classical Legendre-Clebsch condition)}$$

In the case of a singular control, this condition is satisfied trivially, because $\frac{\delta}{\delta u}(\frac{\delta H}{\delta u}) = 0$. For this case, we turn to another more well-suited condition, that looks much like the convexity condition above. The two version of the condition are the following:

Condition for minimality (Bryson and Ho 1969, Bell and Jacobson 1975):

$$(-1)^k \frac{\delta}{\delta u}\left[(\frac{\delta}{\delta t})^{2k}\frac{\delta H}{\delta u}\right] \geq 0$$

Condition for maximality (Sethi and Thompson 1981):

$$(-1)^k \frac{\delta}{\delta u}\left[(\frac{\delta}{\delta t})^{2k}\frac{\delta H}{\delta u}\right] \leq 0$$

This condition is known as the generalized Legendre-Clebsch condition (alternatively, Kelley-Contensou test). The proof of this inequality for the case when $k = 1$ uses second variation of the utility function.

According to Bryson and Ho (Bryson and Ho 1969, 258-261), these following facts always hold:

(i) The variable $2k$ is always even. A problem is a singular problem of order $2k$ if the following hold:

$$(\frac{d}{dt})^i \frac{\delta H}{\delta u} = 0 \text{ for all } i = 0, 1, ..., (2k-1)$$

$$\text{and } (\frac{d}{dt})^{2k}\frac{\delta H}{\delta u} = a(x, \lambda) + b(x, \lambda)u$$

(ii) The singular surface in the (x, λ)-space is of dimension $(2n - 2k)$, where n is the dimension of the state vector x and $2k$ is the order of the singular problem.

4.2. Chattering. Zelikin and Borisov (Borisov and Zelikin, 1994) have studied problems the optimal solutions of which are not piecewise continuous, but merely measurable: these optimal controls have an infinite number of discontinuous jumps over a finite time horizon. This behavior is known as *chattering*. Chattering does not allow for a direct use of Pontryagin's maximum principle due to the fact that it is not considered an admissible policy as it is not piecewise continuous, *i.e.* there does not exist a non-zero-length time interval with a continuous control.

5. Current-value Hamiltonian function

Continuing the discussion started in the second section of this chapter, this section present a revised version of the Maximum Principle. Suppose that the utility function is of the form: $U(t, x, u) = V(t, x, u)e^{-\alpha t}$. Then, we need to revise the conditions of the Maximum Principle to account for this change in the utility function. The Hamiltonian function will be:

$$H(t, x, u, \lambda) = V(t, x, u)e^{-\alpha t} + \lambda f(t, x, u)$$

so we define the current-value Hamiltonian function to be:

$$H_c(t, x, u, \lambda) = V(t, x, u) + \lambda e^{\alpha t} f(t, x, u).$$

Letting $\mu = \lambda e^{\alpha t}$, we rewrite the current-value Hamiltonian function as:

$$H_c(t, x, u, \lambda) = V(t, x, u) + \mu f(t, x, u).$$

Notice that:

$$x' = \frac{\delta H}{\delta \lambda} = f(t, x, u) = \frac{\delta H_c}{\delta \mu}$$

Also,

$$\lambda' = -\frac{\delta H}{\delta x} = \mu' e^{-\alpha t} - \alpha \mu e^{-\alpha t}$$

But,

$$H = H_c e^{-\alpha t} \;\Rightarrow\; -\frac{\delta H}{\delta x} = -\frac{\delta H_c}{\delta x} e^{-\alpha t}$$

So,

$$-\frac{\delta H_c}{\delta x} e^{-\alpha t} = \mu' e^{-\alpha t} - \alpha \mu e^{-\alpha t}$$

$$\mu' = -\frac{\delta H}{\delta x} + \alpha \mu$$

Therefore, the conditions for the revised Maximum Principle are the following:

$$\begin{aligned} &\max_u H_c(t, x, u, \lambda) \text{ for all } t \in [0, T] \\ &x' = \frac{\delta H_c}{\delta \mu}, \\ &\mu' = -\frac{\delta H}{\delta x} + \alpha \mu \\ &\mu(T) e^{-\alpha T} = 0. \end{aligned} \tag{2.3}$$

6. Euler's method of approximation

Euler's method of approximation is a linear one-step approximation method that will be useful later on when we will need to discretize the differential equation of the continuous-time problem. In this section, we first present the method, and then offer a discussion of the magnitude and the bounds of the error from the approximation. Lastly, we will prove that as the step used in the approximation decreases, the approximation becomes more accurate.

Consider the differential equation: $y'(t) = f(t, x(t))$ with initial condition $x(t_0) = x_0$. Then, by choosing a time step h, the discrete form of this equation becomes $x_{n+1} = x_n + hf(t_n, x_n)$, or alternatively, $x(t_0 + h) = x(t_0) + hf(t_0, x(t_0))$. The magnitude of the error produced by this approximation method can be found by comparing this first-order approximation to the Taylor expansion of $x(t_0 + h)$.

$$\text{Euler's method: } x(t_0 + h) = x(t_0) + hf(t_0, x(t_0))$$

$$\text{Taylor's expansion: } x(t_0 + h) = x(t_0) + hx'(t_0) + \frac{1}{2}h^2x''(t_0) + O(h^3)$$

We know that:

$$x''(t_0) = \frac{df(t_0, x(t_0))}{dt} + \frac{df(t_0, x(t_0))}{dx} f(t_0, x(t_0))$$

Hence, Taylor's expansion can be rewritten as:

$$x(t_0+h) = x(t_0)+hx'(t_0)+\frac{1}{2}h^2\left[\frac{df(t_0, x(t_0))}{dt} + \frac{df(t_0, x(t_0))}{dx} f(t_0, x(t_0))\right]+O(h^3)$$

Therefore, the error that Euler's approximation yields is:

$$\frac{1}{2}h^2\left[\frac{df(t_0,x(t_0))}{dt}+\frac{df(t_0,x(t_0))}{dx}f(t_0,x(t_0))\right]+O(h^3).$$

As $h \to 0$, this error approaches zero.

The error bounds are determined by the following inequality:

$$|\epsilon_{n+1}| \leq \frac{hM}{2L}(e^{L(t-t_0)}-1)$$

where ϵ is the error, h is the step size, M is an upper bound on $|x''|$ for all t in the considered time interval, and L is the Lipschitz constant for f.

7. The logistic nature of the SIS dynamic equation

In this section, I will give a short description of the logistic form and the solutions of the following differential equation:

$$x' = \beta x(N-x) - \gamma x = f(x,u).$$

The classical logistic function is of the form:$f(x) = ax(1-\frac{x}{K})$, where K is referred to as the *carrying capacity*, or *saturation level*.

As it can be easily seen, the first part of our differential equation is of such a form: $\beta x(N-x) = \beta N x(1-\frac{x}{N})$. Akin to the analysis of a similar harvesting model (Gordon-Schaefer fishing model) by Clarke (Clarke 1976), we can write the differential equation as:

$$x' = \beta N x(1-\frac{x}{N}) - \gamma x.$$

To obtain the equilibria of this equation, we set $x' = 0$. For any $\gamma < \beta N$, the equilibria is $x = 0$ and $x = K(1 - \frac{\gamma}{\beta})$. It can be shown that the second equilibrium (the nontrivial equilibrium) is asymptotically stable.

7.1. Solutions to the SIS differential equation. We are given the homogeneous quadratic differential equation of the form

$$\frac{dx}{dt} = \beta x(N - x) - \gamma x$$

and we need to find a closed form solution for $x(t)$. By separating the variables t and x, we get:

$$\frac{dx}{x(\beta N - \gamma - \beta x)} = dt$$

Then, integrating both sides:

$$\int \frac{1}{x(\beta N - \gamma - \beta x)} dx = \int dt$$

By the method of partial fractions, we find that:

$$\frac{1}{x(\beta N - \gamma - \beta x)} = \frac{1}{(\beta N - \gamma)x} + \frac{\beta}{(\beta N - \gamma)(\beta N - \gamma - \beta x}$$

Therefore, the left-hand-side integral can be written as:

$$\frac{1}{\beta N - \gamma} \int (\frac{1}{x} + \frac{\beta}{\beta N - \gamma - \beta x}) dx = \frac{1}{\beta N - \gamma} (\ln |x| - \ln |\beta N - \gamma - \beta x|) =$$

$$= \frac{1}{\beta N - \gamma} \ln |\frac{x}{\beta N - \gamma - \beta x}|$$

The integral of the right-hand-side is:

$$\int dt = t + k$$

Therefore,

$$\frac{1}{\beta N-\gamma}\ln|\frac{x}{\beta N-\gamma-\beta x}|=t+k$$

$$(\beta N-\gamma)(t+k)=\ln|\frac{x}{\beta N-\gamma-\beta x}|$$

$$\frac{x}{\beta N-\gamma-\beta x}=\pm e^{(\beta N-\gamma)(t+k)}=Ae^{t(\beta N-\gamma)}$$

where $A=\pm e^{k(\beta N-\gamma)}$. Then, by isolating the terms that contain x in them (note that x is used instead of $x(t)$ out of convenience), we get:

$$x(t)=\frac{(\beta N-\gamma)Ae^{t(\beta N-\gamma)}}{1+\beta Ae^{t(\beta N-\gamma)}}$$

By multiplying both the numerator and the denominator of $x(t)$ by $e^{-(t+k)(\beta N-\gamma)}$, we get:

$$x(t)=\frac{\beta N-\gamma}{\beta+Ae^{-t(\beta N-\gamma)}}$$

Letting $C_1=\frac{A}{\beta N-\gamma}$, we rearrange the denominator and get a solution of the form:

$$x(t)=\frac{\beta N-\gamma}{\beta+C_1(\beta N-\gamma)e^{-t(\beta N-\gamma)}}$$

8. Discretization of the discounting factor

This last section briefly clarifies the discrete-time formulation of the exponential discounting factor used in continuous time. Suppose that we are looking at time $t=T$; the immediate costs incurred at that time are discounted by a factor $e^{-\alpha T}$ back to the present $t=0$. Suppose that we partition this time horizon $[0,T]$ into equal intervals of length δ. Then, we will have $\frac{T}{w}$ such intervals. Consider each of these intervals as one time

period in the discretized problem. What is the appropriate discount factor in this discrete-time framework?

If $w = 1$, then $[0, T]$ will be divided into T time periods, and the discount factor will be of the form: $a = \frac{1}{1+\alpha}$. In the general case when w is not normalized to a length of 1, $a = \frac{1}{1+w\alpha}$. In order to see how is this discrete-time discount factor an approximation of the continuous-time exponential discount factor, we consider the limiting case when $w \to 0$. Viewed from time $t = 0$, the momentary costs of the last period T/w are discounted by a factor of $(\frac{1}{1+w\alpha})^{T/w}$. But, as the length of each period approaches zero, we know that

$$\lim_{w\to 0}(\frac{1}{1+w\alpha})^{T/w} = e^{-\alpha T}.$$

Therefore, the appropriate discrete-time discount factor corresponding to $e^{-\alpha t}$ is $(\frac{1}{1+w\alpha})^{t/w}$.

CHAPTER 3

Essay One:

Attempting to reconcile a classical debate

"It is, in a sense, the single most effective way for the system to grow, so that if we are planning long-run growth, no matter where we start, and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end."

Robert Dorfman, Paul Samuelson, and Robert Solow

"Is the fate of calculus tied to infinitesimals, or must it not be given a rigorous status from the point of view of finite representations? It is precisely this alternative between infinite and finite representation that is at issue when we speak of the 'metaphysics' of calculus."

Gilles Deleuze

This chapter delves into the investigation of potential sources of the discrepancies between the discrete-time and the continuous-time solutions offered by Sanders (Sanders 1971) and Sethi (Sethi 1974) respectively. Extending the discussion begun in the first chapter with the brief description of the argument of each of the articles, the first section of this chapter provides further details on the methods employed by the articles and on a recent attempt of Anderson and Salant to shed light on this controversy. The work of Anderson and Salant opens several research paths that are undertaken and explored in the second and the third section. Section 4 provides a re-evaluation of Sanders' argument by extending the argument to a discrete model with length of period w. The fifth section reconsiders the optimality of the singular solution offered by Sethi by checking whether the Generalized Legendre-Clebsch condition holds for this case. Section 6 concludes the chapter.

1. Expository discussion

Sanders, being the first one to start this scholarly conversation, set up a discrete-time model of an infectious disease with costs that depend linearly in the size of the infected population and in the effort chosen to treat the disease. His model was one of the earliest attempts that preceded the emergence of an entire area of economics of infectious diseases. Therefore, the choice of the SIS model was reasonable because of its simplicity: the population size is fixed and the population consists of only susceptible and infected

individuals. Furthermore, in order to keep things simple, he excluded the possibility of random recoveries from the dynamics of the disease; hence, the only two factors affecting the spread of the infection are the interactions between susceptible and infected individuals, and the treatment policy undertaken by a health agent. The treatment effort, expressed as a portion of the infected individuals chosen to be "removed" of the infected subpopulation, has constant marginal cost and an upper bound due to technological limitations. So, while Sanders was aiming to set up a simple model of infectious diseases, he managed to include three very problematic features in his model, – features that, as it will become apparent later on, account for the atypical solution structure in discrete and continuous time: (i) linear costs, (ii) nonlinear motion law, and (iii) bounded control.

Equation (1.1) presents the discrete-time optimization problem in mathematical form. The corresponding value function equation is the following:

$$V_n(x) = \min \{Cx + K\gamma + \alpha V_{n-1}(x + \beta x(N - x) - \gamma x)\}$$

In the way in which Sanders sets up the problem, there are two facts worth noticing: 1) his interpretation of the upper bound on γ and 2) his generalization of the difference equation for a period of length Δ. As mentioned earlier as well, the treatment parameter γ takes values on an interval $[0, b]$. Sanders interprets b as the treatment success rate, or alternatively, as $1 - T$ where T is the treatment failure rate. This formulation implicitly suggests that b cannot be greater than 1, hence γ takes values between 0 and (at

most) 1. So, the structure of the optimal policy proposed by Sanders crucially depends on the fact that γ can be at most 1. The question of how does the maximum value of b – and more fundamentally, the nature and the interpretation of b – change when going from a discrete-time framework to a continuous-time framework is of particular importance in our analysis. Secondly, Sanders generalizes the first-order difference equation for the infection level in the following form:

$$x(t + \Delta t) = x + \beta \Delta t x(N - x) - \gamma x.$$

While he does not make direct use of this formulation in his article, as he normalizes $\Delta t = 1$, this discrete-time formulation is troublesome because it does not capture the size of the effect of a treatment policy γ for periods of different lengths. In other words, according to this formulation, a certain treatment level, say γ^*, has the same impact $\gamma^* x$ (where x is the infection level at the beginning of the period) for both a time period of length $\Delta t = 0.5$ and $\Delta t = 100$.

The major theorem of the article states that: a) the value function is a monotone increasing function with respect to x, b) it has non-positive second derivatives with respect to γ and negative second derivatives with respect to x, and 3) the optimal policy is $\gamma^* = 0$ or b. From this theorem, several results are concluded: 1) the strict concavity of the value function with respect to x shows that the optimal policy is either 0 or b, so it is never at an intermediate level between 0 and b, 2) for a given infection level x^*, if $\gamma_n^*(x^*) = 0$ for a

certain n, then for any $n' < n$, $\gamma^*_{n'}(x^*) = 0$; if $\gamma^*_n(x^*) = b$ for a certain n, then for any $n' > n$, $\gamma^*_{n'}(x^*) = b$, 3) the health agent *never* treats anybody if $Kb - [\frac{\alpha}{1-\alpha}]CN < 0$, 4) for infinitely long planning horizons (i.e $n \to \infty$), in any period n provided that $\alpha(1 + \beta n) < 1$ (i.e n is not too far ahead in the future), $\gamma^* = 0$ if $Kb - \frac{\alpha Cbx}{1-\alpha(1+\beta n)} > 0$ and $\gamma^* = b$ if $Kb - \frac{\alpha Cbx}{1-\alpha(1-\beta n)} < 0$. The third result raises the question: what is the optimal policy if x is such that $Kb - \frac{\alpha Cbx}{1-\alpha(1+\beta n)} < 0$ and $Kb - \frac{\alpha Cbx}{1-\alpha(1-\beta n)} > 0$? The previous results establish that the optimal policy is always in the boundary of the control range, so although we do not exactly know if $\gamma^* = 0$ or $\gamma^* = b$ for such x, we do know that it will always be extremal. The main proof that Sanders provides will be revisited in the fourth section, when I attempt to extend it for periods of length w instead of of unit length.

Sethi transferred Sanders' model to continuous-time, in the form of the optimization problem presented in Equation (1.2). Unlike in Sanders' work, this article places no restrictions on the value that b can take. Formally speaking, in the case when b is infinite, the control appears to be an impulse control: the agent is able to treat every one in a single instant. This distinction might be rooted in the way in which we think about "action" in discrete and continuous time. The upper bound on the upper bound is crucial to the solutions of the problem. The size of the upper bound b is reflected in the turnpike solutions that Sethi identifies: if b is not large enough, then the turnpike will not be achieved, therefore the optimal policy will be a strictly bang-bang solution.

Because of the linear nature of the costs and the linear dependence of the change in x on the control, the Hamiltonian function in Sethi's problem will be linear in the control:

$$H = -(Cx + K\gamma) + \lambda[\beta x(N - x) - \gamma x] = -Cx + \lambda\beta x(N - x) + (-K - \lambda x)\gamma$$

When the coefficient before the control term in the Hamiltonian function is not zero, the optimal control will be either 0 or b, depending on the sign of that coefficient. But, in the case when this coefficient is equal to zero, Pontryagin's Maximum Principle fails to provide us with an optimal control due to the fact that the second derivative of the Hamiltonian with respect to the control is zero as well. The optimal control for this case should be found by other methods, and its optimality needs to be proven. As a reminder, in the second chapter we introduced the notion of singular controls, which may be either partially singular or totally singular, depending on whether they are optimal over the entire horizon or only several intervals of that horizon. Referring back to one of the earliest articles on the study of the existence of singular controls (Johnson 1963), the authors note that the solution for the type of problem when the Hamiltonian is linear in the control can be of several forms: bang-bang (piecewise-continuous control where every "piece" takes values in the boundary of the control set), chattering (the control is not even piecewise-continuous, only measurable, in which case the control switches infinitely many times within a finite horizon), totally singular, or any finite concatenation of each of these. Hence, unlike Sethi's statement

that "the maximum principle immediately yields the form of the optimal control" (Sethi 1974, 682), the form of the optimal control should be confirmed by tools other than the maximum principle, some of which Sethi uses in his article.

As both the control variable and the state variable are scalars, Miele's method of extremization of line integrals, which is based on an application of Green's theorem, can be employed to find the expression for the singular (or steady-state) control (Miele 1961) . The same control can be identified by taking the time derivatives of the partial derivative of the Hamiltonian function with respect to γ; this argument will be explored in greater detail in Section 5. Sethi uses such an application of Green's theorem to identify an expression for the singular control, and then he employs a switching-point analysis to identify the complete structure of the optimal control, which is a so-called *bang-singular-bang* (or *bang-off-bang*) structure. The solution, hence, turns into totally singular only if $x_0 = x^s$ and the the problem is free-end-point or the fixed end point is $x_f = x^s$, while it turns into strictly bang-bang if the upper bound on control is sufficiently small and/or the optimizing horizon is sufficiently short.

The most recent contribution that attempts to advance the debate between Sanders and Sethi is a paper by Anderson and Salant titled "Hunting Bacteria" (Anderson and Salant, 2011). Anderson and Salant consider the discrete-time problem introduced by Sanders and check the optimality of the turnpike solution identified by Sethi in discrete time. They do so by

assuming that when the level of the infection is at the turnpike level x^s, a small perturbation $\pm h$ from the steady-state policy γ^s should increase the costs of the program as γ^s is the optimal policy. They first find an expression for $x^s = \frac{rK}{C-K\beta}$ and $\gamma^s = \beta(N - x^s)$ which are the discrete-time counterparts of Sethi's x^s and γ^s (r is the discrete-time counterpart of the continuous time discount rate α). Then, the authors suggest a single perturbation $\gamma^s + h$ at period t, where t is assumed to correspond to a moment at which the singular solution is optimal in continuous time. This perturbation will result in an infection level x_{t+1} in the next time period $(t + 1)$, so the infection level will move away from the steady-state level. Further, they assume that at period $(t + 1)$, we apply a correct policy γ_{t+1} that would bring the infection level at period $(t + 2)$ back to x^s. Then, Anderson and Salant show that the two-period costs for this perturbation are lower than the costs of applying Sethi's policy γ^s. In addition, they prove that the cost function is strictly concave in the neighborhood around $h = 0$. This means that any policy that deviates by h from the steady-state solution performs better than the steady-state policy. Therefore, they conclude that in this discrete-time framework, Sethi might have mistaken a local maximum for a global minimum, as any perturbation around $h = 0$ produces lower costs than the case of $h = 0$. This is an important result as it shows the non-optimality of the steady-state control proposed by Sethi in the discrete time framework. Nevertheless, this analysis does not bridge the gap between the

discrete-time framework and the continuous-time framework; it rather reinforces the discrepancies between these two frameworks. Secondly, one might wonder whether the discrete-time steady-state solution that is analogous to the one that Sethi identifies in continuous time is the only candidate for being the supposedly optimal singular solution. The intuitive similarity between discrete-time and continuous-time optimal policies is not the (only) ground for this statement. In fact, Anderson and Salant prove that if there exists a steady-state solution in discrete-time, that solution will be of a similar form to the one that Sethi identifies in continuous time (Anderson and Salant 2011, 2).

These three articles have guided the work that will be presented in the following sections. The questions that we have attempted to answer while examining these articles have been basically two: (i) are there any faulty assumptions, mistakes in the optimization techniques, and/or misinterpretations of results in the analysis of each of these papers, and (2) is there a way to extend the analysis started by Anderson and Salant in continuous time. Finally, throughout the analysis presented in the following sections, we have kept in mind the complexities – both known and unknown to us,– that arise when moving from one time scale to another. Indeed, if there is one thing that this analysis has made us quite aware of, that is the enormous variety of such complexities. In this spirit, Jacobson and Mayne (1970) note the following when attempting to apply their variation-based algorithmic method of differential dynamic programming to discrete-time problems: "In

one respect discrete-time systems are simpler to analyze than the continuous-time systems,[...]– differential equations are replaced by difference equations whose solutions are easier to compute. However discrete-time systems also produce complications of their own. A non-infinitesimal change in the control u_i at time i produces non-infinitesimal changes in the subsequent trajectory. For the continuous-time system, on the other hand, a non-infinitesimal change in the control action $u(t)$ over the interval $[t_1 - \epsilon, t_1 + \epsilon]$ ($\epsilon > 0$ but arbitrarily small) produces small, or order ϵ, variations in $x(t)$ ($t > t_1$)." (Jacobson and Mayne 1970, 99).

2. Perturbation in continuous time

This section discusses two alternative approaches that we have explored aiming to construct an argument that is structurally similar to the one made by Anderson and Salant, but that tackles the problem in continuous time. The first approach will extend the exercise of Anderson and Salant by incorporating the length of the time period w as a variable in the model. This will be achieved by looking at the difference equation in (1.1) as an Euler approximation to the original differential equation in (1.2)[1]. Then, we will attempt to observe the behavior of the h-perturbation as the length of the time period approaches zero. Notice that this exercise slightly modifies the framework of Anderson and Salant, and is still restricted to discrete time

[1]I call the differential equation "original" because the Kermack-McKendrick model was at first formulated as a differential equation rather than a difference equation.

only. The second approach will replicate the exercise proposed by Anderson and Salant in continuous time, by distinguishing between two cases: 1) when a perturbed policy $\gamma^s + h$ is applied for an interval of length Δt and then a restorative policy γ_{t+1} is applied during a second time interval of the same length, and 2) when Δt is approaching zero in the previous scenario. This second approach asks in essence: if we perturb the steady-state policy for a single instant (whatever that means mathematically), and then correct that perturbation in the next instant (a strategy parallel to the one that Anderson and Salant employ), would we have just lowered the costs by doing this?

2.1. Extension of "Hunting Bacteria" in discrete-time. Following Euler's method of approximation of differential equations, the difference equation corresponding to $\frac{dx}{dt} = \beta x(N - x) - \gamma x$ is:

$$x_{t+1} = x_t + w(\beta x(N - x) - \gamma x) \tag{3.1}$$

where w denotes the length of the time period. Notice that when we let $w = 1$, we return to the standard case discussed in "Hunting Bacteria". Based on this discrete-time dynamics, we will rewrite the expressions for x^s, γ^s, x^{t+1}, and γ_{t+1} following an argument parallel to the one made in the "Hunting Bacteria" piece. The discrete-time cost function that accounts for the length of the time period will be of the form:

$$\sum_{t=1}^{T} (\delta(w))^t (Cwx_t + Kw\gamma_t) \tag{3.2}$$

The discount rate δ varies for different time period lengths, therefore δ is a function of w. For a general period length w, the discounting factor will be of the form:

$$\delta = \frac{1}{1+wr} \tag{3.3}$$

Refer to the discussion of the discounting factor in Chapter 2 for further details.

Using Lagrange multipliers to minimize the cost function, we let the Lagrangian be:

$$L = \sum_{t=1}^{T} (\delta(w))^t [(Cwx_t + Kw\gamma_t) + \lambda_t(x_t + w\beta x_t(N - x_t) - w\gamma_t x_t - x_{t+1})] + \lambda_0(\bar{x} - x_1)$$

An interior solution will satisfy the first-order conditions: $L_{\gamma_t} = 0$ and $L_{x_t} = 0$ for all $t = 1, ..., T$. Therefore, from the first-order condition with respect to γ_t, we derive that:

$$L_{\gamma_t} = \delta^t(Kw - w\lambda_t x_t) = 0$$

$$\lambda_t = \frac{K}{x_t} \tag{3.4}$$

The first-order condition with respect to x_t is:

$$L_{x_t} = \delta^t[Cw + \lambda_t + \lambda_t w\beta N - 2\lambda_t w\beta x_t - \lambda_t w\gamma_t] - \lambda_{t-1}\delta^{t-1} = 0$$

$$\delta(Cw + \frac{K}{x_t}(1 + w\beta N - 2w\beta x_t - w\gamma_t)) - \frac{K}{x_{t-1}} = 0 \tag{3.5}$$

The turnpike infection level that Sethi suggests is stationary, therefore, in order for it to be maintained from one period to another, the number of the newly infected people should be equal to the number of the people that were effectively treated during the last period. That means that $w(\beta x^s(N - x^s) - \gamma^s x^s) = 0 \Rightarrow \gamma^s = \beta(N - x^s)$. Also, we know that this steady state should satisfy equations (1.8) and (1.9) derived above. Therefore,

$$\delta(Cw + \frac{K}{x^s}(1 + w\beta N - 2w\beta x^s - w\gamma^s)) - \frac{K}{x^s} = 0$$

where $\gamma^s = \beta(N - x^s)$. Therefore, we can derive an expression for x^s:

$$x^s = \frac{(1-\delta)K}{w\delta(C - K\beta)} \tag{3.6}$$

Suppose that the steady-state level of treatment effort γ^s is perturbed to $\gamma^s + h$. The infection level in the next period will be x_{t+1}, such that:

$$x_{t+1} = x^s + w\beta x^s(N - x^s) - w(\gamma^s + h)x^s$$

Hence,

$$x_{t+1} = x^s(1 - wh) \tag{3.7}$$

Starting from an infection level x_{t+1}, we can find γ_{t+1} that would restore the infection level at x^s in the next period. The difference equation for the next time period is:

$$x^s = x_{t+1} + w\big(\beta x_{t+1}(N - x_{t+1}) - \gamma_{t+1}x_{t+1}\big)$$

Substituting for x_{t+1}, we can solve for γ_{t+1} [2] :

$$\gamma_{t+1} = \frac{h}{hw-1} + \beta\big(N - x^s(1-wh)\big) \tag{3.8}$$

Having obtained expressions for x^s, γ^s, x_{t+1}, and γ_{t+1}, we can compute the costs of this perturbation over the two time periods:

$$\text{Costs} = \delta^t\Big[wCx^s + wK(\gamma^s + h)\Big] + \delta^{t+1}\Big[wCx_{t+1} + wK\gamma_{t+1}\Big]$$

Dividing by δ^t and simplifying the expression further, we get:

$$\text{Costs} = \Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)} + wK\big(\beta(N-x^s)+h\big)\Big] + \\ \delta\Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)}(1-wh) + wK\Big(\frac{h}{hw-1} + \beta\big(N - x^s(1-wh)\big)\Big)\Big]$$

For a fixed value of w, we could rearrange the terms by isolating all the terms that include h in them and labeling all the other terms as constants. Therefore, the two-period cost function is:

$$\text{Costs} = \textit{constants} + wK\Big[h\big(1 - C\frac{1-\delta}{C-K\beta} + \beta w x^s\big) + \frac{h}{hw-1}\Big] \tag{3.9}$$

When we differentiate this cost function twice and we evaluate the value of the second derivative at $h = 0$, we get the expression:

$$wK\left(w\left(1 - 2\left(1 + w\right)\right)\right) \tag{3.10}$$

As the value of w approaches zero, the value of this second derivative approaches zero as well, so the concavity of the function in the neighborhood

[2] For further details on the computations of this subsection, please see Appendix A

around $h = 0$ remains inconclusive. We are interested to know what happens to the values of x^s,γ^s, x_{t+1}, γ_{t+1} and the two-period cost function as the length of the time periods shrinks infinitely, *i.e.* when $w \to 0$. Our ultimate goal is to be able to state whether the cost function is concave around the neighborhood of $h = 0$ as w approaches 0. When the length of the time period approaches zero, we obtain the following limits for x^s,γ^s, x_{t+1}, γ_{t+1}, which are in agreement with what we would expect intuitively:

$$\lim_{w \to 0} x^s = \frac{K\alpha}{C - K\beta}$$

$$\lim_{w \to 0} \gamma^s = \beta(N - \frac{K\alpha}{C - K\beta})$$

$$\lim_{w \to 0} x_{t+1} = \frac{K\alpha}{C - K\beta}$$

$$\lim_{w \to 0} \gamma_{t+1} = -h + \beta N - \frac{K\alpha}{C - K\beta}\beta$$

When $w = 1$, it is easily showed that the cost function is strictly concave in h when $h = 0$. For a more general case of a time period length w, as we showed above, the cost function becomes more complicated due to the w factor in front of all the terms that contain h in that function. In other words, as we let $w \to 0$, so as we let the time period become infinitesimally small, all the terms with h in them disappear from the expression, and we are left with no clue on what the concavity of that cost function is around $h = 0$.

While this exercise provides an elegant extension of the discrete-time framework that Anderson and Salant have constructed, it yields no specific

conclusions on whether the results that they find hold even when we shrink the period length to zero. At this point, it is reasonable to conclude that any discrete-time method that relies on this shrinking strategy in order to connect to the continuous-time framework will most likely fail due to difficulties that arise in the limit. In the next subsection, we turn to a method of carrying the argument made in "Hunting Bacteria" over to a continuous-time framework. This method aims to get closer to the continuous-time terrain in which Sethi's work lives.

2.2. Continuous-time "Hunting Bacteria".

2.2.1. *When the length of the time interval is not approaching zero.* Let's suppose that at time t_1, the level of the infection level is at the turnpike level x^s and we choose to perturbate the treatment effort from the turnpike level γ^s to $(\gamma^s + h)$. Analogously to the reasoning in the discrete-time model, this new level of treatment effort will affect the infection level at time t_2 (t_2 chosen such that $t_2 > t_1$ and $t_2 - t_1 = \Delta t$). Then, we will need to apply a treatment effort γ_{t+1} at ${t_2}$[3] in order for the infection level at time t_3 to return back to the stationary level x^s (t_3 chosen such that $t_3 > t_2$ and $t_3 - t_2 = \Delta t$). During the time interval $[t_1, t_2]$, the infection level changes according to the differential equation:

$$x_1' = \beta x(N - x) - (\gamma^s + h)x \tag{3.11}$$

[3]This notation is chosen in order to keep up with the discrete-time notation introduced in Anderson and Salant.

with initial condition $x_0 = x^s$. Similarly, the infection level during the time interval $[t_2, t_3]$ is governed by the differential equation

$$x_2' = \beta x(N - x) - (\gamma_{t+1})x \tag{3.12}$$

with initial condition $x_0 = x(t_2)$ (and terminal condition $x(t_3) = x^s$). Our goal is to calculate the cost function for the time interval $[t_1, t_3]$ and examine whether and how this cost function depends on h.

First, we evaluate the steady-state level of infection and the steady-state treatment level:

$$x^s = \frac{K\alpha}{C - K\beta}$$

$$\gamma^s = \beta(N - x^s) = \beta(N - \frac{K\alpha}{C - K\beta})$$

The closed-form solutions for the differential equations mentioned above are:

$$x_1(t) = \frac{\beta N - \gamma^s - h}{\beta + C_1\beta N e^{-t(\beta N - \gamma^s - h)} - C_1(\gamma^s + h)e^{-t(\beta N - \gamma^s - h)}} \tag{3.13}$$

for equation (3.11) and:

$$x_2(t) = \frac{\beta N - \gamma_{t+1}}{\beta + C_2\beta N e^{-t(\beta N - \gamma_{t+1})} - C_2\gamma_{t+1}e^{-t(\beta N - \gamma_{t+1})}} \tag{3.14}$$

for equation (3.12).

Notice that the initial condition for the first differential equation is $x_1^0 = x^s$, while the initial condition for the second differential equation is $x_2^0 =$

x_{t+1}. Hence, we can solve for C_1 and C_2 based on these initial conditions. Using these initial conditions, we can rewrite the equations in such a form that they include γ_{t+1} in them; the goal is to isolate an expression for γ_{t+1} so that we can then evaluate the costs for this perturbation and analyze its dependence on h. After several algebraic manipulations[4], we receive the following:

$$\beta N - x^s\beta = \gamma_{t+1} + x^s\big(e^{-\Delta t(\beta N-\gamma_{t+1})}(C_2\beta N - C_2\gamma_{t+1})\big) \tag{3.15}$$

It is mathematically intractable to isolate γ_{t+1} from the equation above, as γ_{t+1} appears in the exponential terms and in the linear terms of the equation. This obstacle does not allow us to talk any further about the costs of an h-perturbation over non-shrinking time intervals. At this point, we consider the limit of equation (3.15) to overcome this difficulty, thus turning our attention to the case of shrinking time intervals.

2.2.2. *When the length of the time interval is approaching zero.* Building upon the results of part a, we take the limit of equation (3.15) as $\Delta t \to 0$:

$$\lim_{\Delta t\to 0}(\beta(N - x^s)) = \lim_{\Delta t\to 0}\big(\gamma_{t+1} + x^s\big(e^{-\Delta t(\beta N-\gamma_{t+1})}(C_2\beta N - C_2\gamma_{t+1})\big)\big)$$

$$\Downarrow$$

$$\beta(N - x^s) = \gamma_{t+1} + x^s(C_2\beta N - C_2\gamma_{t+1})$$

$$\Downarrow$$

[4]See Appendix B for a thorough mathematical discussion of this case.

$$\gamma_{t+1} = \frac{\beta N - x^s \beta - x^s C_2 \beta N}{1 - C_2 x^s} \tag{3.16}$$

At this point, we have determined the treatment level that needs to be applied at t_2 in order to return to the stationary level of infection x^s. The cost function for the interval $[t_1, t_3]$ can be written as a sum of costs of the intervals $[t_1, t_2]$ and $[t_2, t_3]$. Hence, the total costs for $[t_1, t_3]$ are:

$$\int_{\Delta t}^{0} \delta^{t_1+\Delta t}\big(Cx_1(t) + K(\gamma^s + h)\big) + \int_{\Delta t}^{0} \delta^{t_1+\Delta t+\Delta t}\big(Cx_2(t) + K(\gamma_{t+1})\big) \tag{3.17}$$

We are interested to evaluate this expression at $\Delta t \to 0$.

Two difficulties that can be noticed this point of our analysis make the approach of alternating between $\gamma^s + h$ and γ_{t+1} not very promising for further results. First, each of the cost integrals cannot be evaluated analytically, so there is no analytical way to compute the antiderivative of the integral. As this problem arises because of the form of the cost function, we expect such a difficulty to arise repeatedly in similar attempts. The second obstacle, which is closely tied to the first one, arises out of the fact that if we attempt to take the limit of the integrals in equation (3.16) without finding an antiderivative expression first, then based on the Fundamental Theorem of Calculus, the sum of integrals goes to zero. This is equivalent to the fact that the instantaneous costs are zero.

3. Green's theorem and the Legendre-Clebsch condition

Before introducing the results of this exercise, it would be worthwhile to point out two important features of the optimization technique that is based on Green's theorem and quickly clarify how do the solutions identified by Sethi using this technique look graphically. First, this technique – although quite limited in its applicability due to the requirement that both the state variable and the control variable ought to be scalars – provides necessary and sufficient conditions for minimality (Leitmann 1967, 64). Secondly, this approach identifies global, rather than just local, optimal solutions (Pontani and Teoffilato 2009, 3). These two features, combined together, assure us that the singular control that Sethi identifies through this technique is indeed globally optimal. Lastly, as we stated in the introduction of this technique in the second chapter, the sign of the integrand inside the double integral is very important when analyzing whether the identified extremum is a maximum or a minimum. Figure 3.1 and 3.2 below show graphically how the sign of the integrand $\frac{K\alpha}{x} - C + K\beta$ changes as shown (both cases of $x^s < 0$ and $x^s > 0$, where x^s is the singular state path graphed in the plane (t, x)-plane, are considered).

This section reconsiders the results obtained by Sethi through the application of Green's theorem by taking the approach of finding the time derivatives of the switching function. The same expression for the singular control path and the singular state path are obtained, but we go one

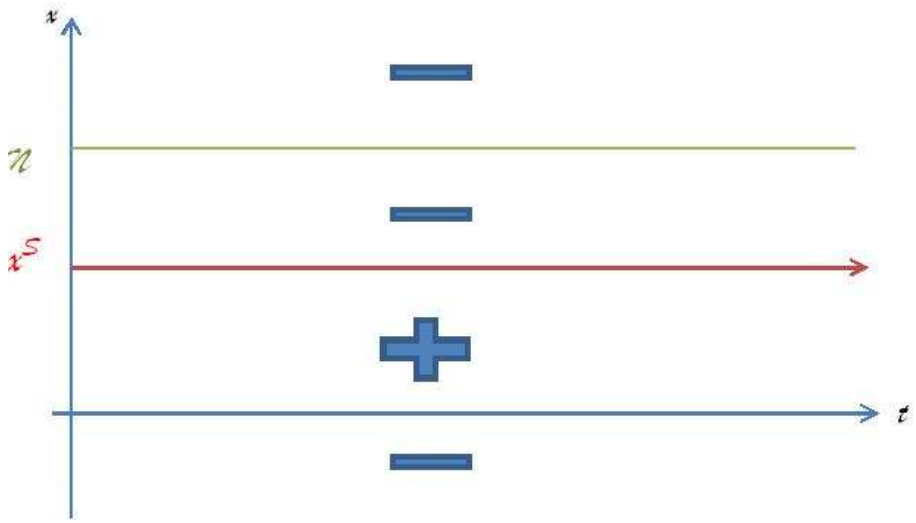

FIGURE 3.1. The sign of the integrand when $x^s > 0$

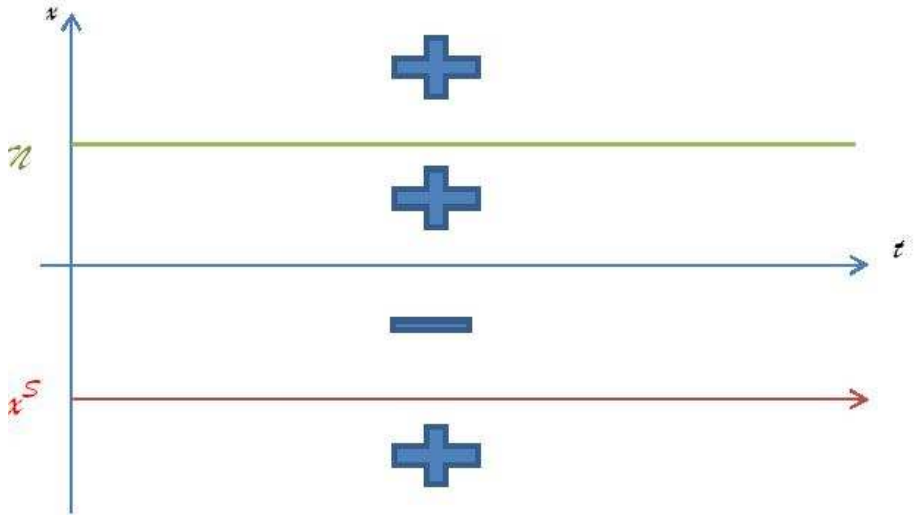

FIGURE 3.2. The sign of the integrand when $x^s < 0$

step further and check whether the Generalized Legendre-Clebsch condition

holds along the singular control.[5] The results confirm the optimality of the singular control.

In the optimal control problem formulated by Sethi, the objective function is linear in the control variable, the differential equation governing the changes in the infection level is nonlinear in the control, and there are inequality constraints in both the state variable and the control variable. This specific class of optimal control problems is best handled by finding an expression for the switching manifold, and then determining the behavior of the optimal path when approaching this manifold. This behavior is determined by the form of the switching function and its higher derivatives. If all its derivatives vanish at a certain time t, then the system has an optimal control of a singular form. Otherwise, the control has just a switch at time t, which means that the value of the control changes from one bound of the control to the other, while the switching function changes sign. Sethi makes use of the switching-point analysis in finding that a three-piece concatenation of the singular control and the bang-bang control is optimal, while he does not employ this analysis in identifying the singular control.

[5]The work of Ledzewicz and Schättler (2002), which contains a successful implementation of the Generalized Legendre-Clebsch condition to confirm the nonoptimality of the singular control in a model of cancer chemotherapy, inspired us to take a switching function analysis and apply the Generalized Legendre-Clebsch condition to Sethi's problem. Later on, the need for a deeper technical understanding of this type of analysis guided us to classical sources such as Bryson and Ho (1969), Bell and Jacobson (1975) etc.

In order to identify a singular control, one needs to take time derivatives of the switching function and set them to zero until the control variable reappears in the derivative, and then solve for that control variable; the obtained expression will be the singular control (Borisov and Zelikin 1994, Bryson and Ho 1975). Then, one needs to check whether this singular control satisfies the Generalized Legendre-Clebsch condition (alternatively known as Kelley's condition) in order to determine whether this singular control path is optimal. These are the steps that I will follow in the rest of this exercise.

Let the Hamiltonian function be:

$$H = -(Cx + K\gamma) + \lambda[\beta x(N - x) - \gamma x] = -Cx + \lambda\beta x(N - x) + \gamma(-K - \lambda x)$$

We also know that:

$$\lambda' = \lambda(\alpha - \beta N + \gamma) + 2\beta x\lambda + C$$

So, the switching function is:

$$\frac{dH}{dt} = W(t) = -K - \lambda(t)x(t)$$

Setting it to zero,

$$\lambda x = -K$$

Taking the first derivative of $W(t)$ with respect to t,[6]

$$W'(t) = -Cx - \lambda\beta x^2 - \lambda\alpha x = -Cx + K\beta x + K\alpha = 0$$

Solving for x^s,

$$x^s = \frac{K\alpha}{C - K\beta}$$

The control variable does not appear in the first derivative of the switching function, so we need to evaluate its second derivative and solve for the singular control:

$$W''(t) = (-Cx+K\beta x+K\alpha)' = -Cx'+K\beta x' = (K\beta-C)(\beta x(N-x)-\gamma x) = 0$$

$$W''(t) = K\beta^2 xN - C\beta xN - K\beta^2 x^2 + C\beta x^2 + \gamma(Cx - K\beta x)$$

Separating γ from the last equation, we get:

$$\gamma = \frac{C\beta(N-x) - \beta^2 K(N-x)}{C - K\beta} = \beta(N-x)$$

which is the expression that Sethi [3] gets for the steady-state solution as well.

In our problem, we are minimizing the discounted costs $(Cx+K\gamma)$, so we are maximizing its negative $-(Cx + K\gamma)$. The Legendre-Clebsch condition for maximality is the following (Sethi 1981):

$$(-1)^k \frac{\delta}{\delta u}\left[(\frac{\delta}{\delta t})^{2k}\frac{\delta H}{\delta u}\right] \leq 0$$

[6]In searching for connections between Miele's method and the switching-point analysis, it is worth noticing that $W'(t)$ is equal to the integrand inside the surface integral in Green's theorem.

So, the singular control identified above is optimal when:

$$\frac{\delta}{\delta\gamma}\frac{d^2W}{dt^2}(\lambda(t), x^s(t)) \geq 0$$

$$(C - K\beta)x^s \geq 0$$

From the expression for x^s, this inequality is equivalent to:

$$K\alpha \geq 0.$$

But $K \geq 0$ (treatment is costly) and $\alpha > 0$ (by assumption), therefore the singular control is optimal.

4. Pulsing over unequal intervals

Having encountered multiple difficulties with the previous approaches in which we alternate γ between $\gamma^s + h$ and γ_{t+1}, we turn our attention to a different kind of pulsing. In this section, we consider the strategy of pulsing between minimal and maximal treatment effort over unequal time intervals. This method is inspired by the solution that Sanders provides for his discrete-time problem. Intuitively, if Sanders correctly claims that it is optimal to alternate between $\gamma = 0$ and $\gamma = b$ depending on the current infection level, then we could think of the time periods of his model as very very small, and hence extrapolate from his work that starting from x^s, it is optimal to chatter between effort levels of $\gamma = 0$ (no effort at all) and $\gamma = b$ in continuous time. Part 1 of this section focuses on laying out an analytical framework for this exercise and identifying several difficulties of approaching

this method analytically. Part 2 presents a numerical example that makes use of this analytical framework and concludes a very important result. Then, this numerical example is extended over an infinite-time horizon to draw some useful economic implications.

4.1. An analytical discussion. In this first part, we will evaluate the overall costs of applying a policy treatment $\gamma = 0$ over a fixed interval $[0, t_1]$ and $\gamma = b$ over a second interval $[t_1, t_2]$ that is long enough to restore the steady-state infection level at time t_2 (we need to keep in mind that $\gamma \in [0, b]$). Then, we will compare these costs to the costs of applying the steady-state treatment policy γ^s over the entire time interval $[0, t_2]$.

For a fixed time interval $[0, t_1]$ during which we follow a minimum treatment policy $\gamma = 0$, the infection level $x_1(t)$ is governed by the differential equation

$$x_1' = \beta x_1(N - x_1) \qquad \text{with initial condition} \qquad x_1(0) = x^s$$

The closed form solution for this initial value problem is:

$$x_1(t) = \frac{\beta N}{\beta + C_1 \beta N e^{-\beta N t}} \qquad \text{where} \qquad C_1 = \frac{N - x^s}{N x^s}$$

At the fixed endpoint t_1 of the interval, the value of the infection level is:

$$x_1(t_1) = \frac{\beta N}{\beta + C_1 \beta N e^{-\beta N t_1}} = \frac{N}{1 + C_1 N e^{-\beta N t_1}}$$

This will be the initial condition for the differential equation that governs the second time interval $[t_1, t_2]$. Hence, the infection level $x_2(t)$ over the

second time interval, during which we are applying a treatment effort $\gamma = b$ is driven by the differential equation

$$x_2' = \beta x_2(N - x_2) - bx_2 \qquad \text{with} \qquad x_2(0) = x_1(t_1)$$

The closed form solution for this initial value problem is:

$$x_2(t) = \frac{\beta N - b}{\beta + C_2 e^{-(\beta N - b)t}(\beta N - b)}$$

where the value of C_2 is determined by the following equation:

$$x_2(0) = \frac{\beta N - b}{\beta + C_2(\beta N - b)} = \frac{N}{1 + C_1 N e^{-\beta N t_1}} = x_1(t_1)$$

Notice that we are considering t_1 to be the initial moment for the second interval. We will account for this by discounting appropriately when calculating the costs. Also, we want $x_2(t_2 - t_1) = x_2(\Delta t) = x^s$, therefore, we set up the following equation:

$$\frac{\beta N - b}{\beta + C_2 e^{-(\beta N - b)\Delta t}(\beta N - b)} = \frac{K\alpha}{C - K\beta}$$

From here, we can evaluate the length of the second interval that restores the steady-state level of infection:

$$\Delta t = -\frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_1(\beta N - b)}\right)}{\beta N - b}$$

So, the length of the second time interval is: $t_1 + \Delta t = t_1 - \frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_1(\beta N - b)}\right)}{\beta N - b}$.

The cost function for $[0, t_1]$ is:

$$\int_0^{t_1} e^{-\alpha t}(C\frac{\beta N}{\beta + C_1 \beta N e^{-\beta N t}})$$

while the cost function for $[t_1, t_1 + \Delta t]$ is:

$$\int_0^{-\frac{\log\left(\frac{\beta N-\beta x^s-b}{x^s C_1(\beta N-b)}\right)}{\beta N-b}} e^{-\alpha(t_1+t)}(C\frac{\beta N-b}{\beta+C_2e^{-(\beta N-b)t}(\beta N-b)}+Kb)$$

Therefore, the overall costs for alternating between $\gamma = 0$ and $\gamma = b$ over $[0, t_2]$ are:

$$\int_0^{t_1} e^{-\alpha t}(C\frac{\beta N}{\beta+C_1\beta Ne^{-\beta Nt}})+\int_0^{-\frac{\log\left(\frac{\beta N-\beta x^s-b}{x^s C_1(\beta N-b)}\right)}{\beta N-b}} e^{-\alpha(t_1+t)}(C\frac{\beta N-b}{\beta+C_2e^{-(\beta N-b)t}(\beta N-b)}+Kb)$$

The cost of applying $\gamma = \gamma^s$ throughout $[0, t_2]$ is:

$$\int_0^{t_1-\frac{\log\left(\frac{\beta N-\beta x^s-b}{x^s C_1(\beta N-b)}\right)}{\beta N-b}} e^{-\alpha t}(Cx^s + K\gamma^s)$$

We need to compare:

$$\int_0^{t_1} e^{-\alpha t}(C\frac{\beta N}{\beta+C_1\beta Ne^{-\beta Nt}})+\int_0^{-\frac{\log\left(\frac{\beta N-\beta x^s-b}{x^s C_1(\beta N-b)}\right)}{\beta N-b}} e^{-\alpha(t_1+t)}(C\frac{\beta N-b}{\beta+C_2e^{-(\beta N-b)t}(\beta N-b)}+Kb)$$

with

$$\int_0^{t_1-\frac{\log\left(\frac{\beta N-\beta x^s-b}{x^s C_1(\beta N-b)}\right)}{\beta N-b}} e^{-\alpha t}(Cx^s + K\gamma^s)$$

which proves mathematically challenging. For this reason, we turn to a numerical example to compare these two costs.

4.2. A numerical simulation. [7]

The initial goal of this exercise (which was undertaken before we took a deeper look in the optimization approach centered around Green's theorem)

[7]The software Maple with 100 digits of accuracy was used for the computations of this example. Maple was used also for most of the complicated algebraic manipulations throughout this chapter.

was to attempt to prove Sethi's proposed optimal solution wrong by use of a numerical counterexample. Indeed, we found one, which in light of what we know now about the effectiveness of Miele's technique, stands out as quite odd. We present this (most likely anomalous) numerical result in this subsection, kindly inviting the reader to identify any potential errors in the procedure. We show that, starting from the turnpike infection level, the costs of applying $\gamma = 0$ for a fixed interval $[0, t_1]$ and $\gamma = b$ for an interval long enough to restore the steady-state infection level x^s are lower than the costs of applying γ^s throughout these two intervals. Sethi argues that, starting from an infection level x^s, it is always optimal (*i.e.* cost-minimizing) to follow a treatment policy γ^s and remain in the turnpike level x^s as long as possible. Clearly, this result should hold for any value of the parameters of the population (β and N) and of the cost function (δ, C, and K). We will show that for a set of values for the parameters of the model, applying γ^s is not optimal. Notice that the problem is approached in continuous time (so we are not discretizing the time in unequal periods of length t_1 and $t_2 - t_1$, but rather applying $\gamma = 0$ repeatedly over $[0, t_1]$ and applying $\gamma = b$ repeatedly over $[t_1, t_2]$, where t_2 is dependent on t_1).

Let $\beta = 0.2$, $N = 100$, $\delta = 0.4065696597...$ (which corresponds to the value $\alpha = 0.9$ in Sethi's specification of the cost function, knowing that $e^{-\alpha} = \delta$), $C = 2.5$, $K = 3.5$, $b = 25$. The first step in this analysis is to

evaluate x^s and γ^s for these parameters. So,

$$x^s = \frac{K\alpha}{C - K\beta} = \frac{0.9 \cdot 3.5}{2.5 - 0.2 \cdot 3.5} = 1.75$$

The corresponding γ^s is:

$$\gamma^s = \beta(N - x^s) = 0.2 \cdot (100 - 1.75) = 19.65$$

We choose the length of the first time interval to be $t_1 = 0.00001$, and then solve for the length of the second time interval following the reasoning presented in Section 3.1. A thorough discussion of the numerical steps followed to reach the results that we are about to introduce can be found in Appendix C. We found that the difference between the cost of applying $\gamma^s = 19.65$ and the cost of alternating between $\gamma = 0$ and $\gamma = 25$ is:

$$6.11 \cdot 10^{-10}$$

Therefore, once we are in the turnpike, it is better to alternate between $\gamma = 0$ and $\gamma = 25$ over carefully-chosen intervals (where the first interval is 0.00001 units of time long, while the second interval is long enough to restore the stationary level of infection $x^s = 1.75$ than to apply a steady-state treatment effort $\gamma^s = 19.65$ [8].

Now, we will extend the result obtained from the previous numerical example to the infinite horizon. Let the set of two adjacent intervals over which we first apply $\gamma = 0$ and then $\gamma = b$ compose a *phase*. Then, the

[8]Notice that $b > 1$ and $\gamma^s > 1$ represent instantaneous intensity efforts, therefore they can take values greater than 1.

numerical example that we have already worked out considers one single phase; our task is to extend the same strategy over an infinite number of phases. A phase starts with the steady-state infection level x^s, and it ends with the same steady-state infection level. The infection level is perturbed in between intervals within a phase, but then it is restored to x^s during the second time interval of the phase. Therefore, the phase length and the costs for one single phase are the same for all the phases over the infinite-time horizon. The only difference is that, from one's perspective at $t = 0$, the costs of later phases are discounted more heavily than the costs of earlier phases. Given that we know the length of one phase, we can say that at $t = 0$, the costs of the second phase are discounted by a factor of $e^{-\alpha\Delta t}$, the costs of the third phase are discounted by a factor of ($e^{-2\alpha\Delta t}$, and so on, the costs of the n^{th} phase are discounted by a factor of $e^{-\alpha\Delta t(n-1)}$, where Δt denotes the phase length. Similarly, it can be argued that the costs of the steady-state strategy that Sethi proposes can be discounted in the same way.

In the numerical example, we have shown that the difference between the costs of these different strategies is $6.11 \cdot 10^{-10}$. When calculating the infinite-horizon difference between these costs, we notice that:

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-\alpha\Delta t(n-1)}\text{Costs/phase} &= \text{Costs/phase}\sum_{n=1}^{\infty} e^{-\alpha\Delta t(n-1)} \\ &= (\text{Costs/phase}) \cdot \frac{1}{1-e^{-\alpha\Delta t}}\end{aligned}$$

Hence, the difference should be weighted by a factor of $\frac{1}{1-e^{-\alpha\Delta t}}$. In the numerical example,

$$(6.11\cdot 10^{-10})\cdot\frac{1}{1-e^{-\alpha\Delta t}} \approx 0.000014528631272132107723838190808570360240128981$$

$$1169220194494036177228203861130780460331178965005819843 1...$$

This is the difference between the costs of the two strategies applied over an infinite-time horizon. In other words, this is by how much Sethi's strategy is more costly than the strategy that we propose in this example if we keep applying these treatment levels forever. If we were to assign units to the values of the parameters that we have been working with so far, we could say that the total costs in the numerical example are in terms of thousands of dollars, then the difference between the costs of the strategies is approximately 14 cents. If the units were millions of dollars, the difference would be about 14 dollars.[9]

Note: Appendix D contains two more attempts that we undertook while struggling to understand the chattering policy and its advantages compared to the turnpike policy. We saw it reasonable not to include this material in our main discussion, but it might be useful to present these efforts for the sake of completeness.

[9]Note that for this numerical example, the population was very small ($N = 100$). The difference in the costs becomes even more significant as we consider larger populations. Also, note that if we let the units of time be years, then these are costs per year. It would follow that in the numerical example: $t_1 = 0.00001$ years $= 5.256$ minutes.

5. An w-extension of the dynamic programming argument

In this section, a reevaluation of the theorem provided in Sanders' work and presented in the introductory discussion of this chapter will be undertaken. More specifically, we will write the difference equation in its general form for the period of length w, and then we will investigate whether the same results that Sanders establishes in the theorem still hold.

This exercise will attempt to mimic the proof of the weak concavity on γ and the strict concavity on x of the value function provided in Appendix A in the article by Sanders (Sanders 1971, 889). The proof will take the length of the time period to be $w = \frac{1}{n}$. Alternatively, suppose we start with a single period problem where $w = 1$, and suppose we divide that time period in n smaller intervals. Then, we make all the necessary changes in the cost function, the difference equation, and the bounds of the treatment level so that they will reflect the change in the length of the time period. Section 2 of this chapter has given a discussion of these changes; that section has modified the control term from γ to $w\gamma$. In this section, we will keep γ as it is in the formulation with $w = 1$, but we will change the upper bound of the control from b to $\frac{b}{n}$. This different formulation resembles Sanders' suggestion about the equation for $x(t + \Delta t)$ (discussed in Section 1), while making the necessary changes in the control set. The major goal of the exercise is to check that the optimal solution is never an interior solution.

If $w = \frac{1}{n}$, then the current-period utility function would be of the form: $U(x, \gamma) = \frac{C}{n}x + K\gamma$, where $\gamma \in [0, \frac{b}{n}]$. The difference equation governing the changes in the infection level is: $x(t + \frac{1}{n}) = x(t) + \frac{\beta}{n}x(N - x) - \gamma x$.

Knowing that $V_0(x) = 0$ (where V_0 is the value function at the end of the last period), $V_1(x) = \frac{C}{n}x$. The condition $V_0(x) = 0$ is analogous to a transversality condition: there is no return from spending in treatment in the last period, as it takes one period for treatment to become effective. Then,

$$V_2(x, \gamma) = \min_{\gamma}\{\frac{C}{n}x + K\gamma + \alpha\frac{C}{n}(\frac{\beta}{n}x(N - x) - \gamma x)\} \qquad \text{where} \qquad \gamma \in [0, \frac{b}{n}].$$

So,

$$\frac{dV_2(x, \gamma)}{d\gamma} = K - \frac{\alpha C x}{n}$$

and

$$\frac{d^2V_2(x, \gamma)}{d\gamma^2} = 0.$$

$V_2(x, \gamma)$ is twice differentiable and its second derivative is non-positive, therefore the function is (weakly) concave with respect to the control. Also,

$$\frac{dV_2(x, \gamma)}{dx} = \frac{C}{n} + \alpha\frac{C\beta N}{n^2} - 2\alpha\frac{C\beta x}{n^2}.$$

So, for $x < \frac{n}{2\alpha\beta} + \frac{N}{2}$, $V_2(x, \gamma)$ is increasing in x. Notice that as $n \to \infty$, $V_2(x, \gamma)$ will be strictly increasing for the entire domain of x. Taking the second derivative with respect to x:

$$\frac{dV_2^2(x, \gamma)}{dx^2} = -2\alpha\frac{C\beta}{n^2} < 0$$

So, $V_2(x, \gamma)$ is strictly increasing and strictly concave on x. By induction, we repeat the reasoning for all the previous minimizing value functions up to $V_n(x, \gamma)$.

Therefore, we see that the same results that Sanders concludes in his work carry over to a framework with discrete time periods of length w. The value function is increasing in x, and concave in both x and γ. Hence, the optimal control is either $\gamma = \frac{b}{n}$ or $\gamma = 0$ for all n. In any discrete-time model – no matter what the length of a certain period – the optimal policy is a strictly bang-bang policy.[10]

6. Conclusions: Persistence of a (better-defined) puzzle

In conclusion, we would like to summarize the attempts that we have made in our quest for the sources of discrepancies between discrete-time and continuous-time optimal solutions for a specific SIS model with linear costs and nonlinear dynamics. Our efforts have been channeled in four major directions: 1) evaluation and generalization of the discrete-time dynamic programming approach taken by Sanders to frameworks of time periods of general length w, 2) validity check of the continuous-time optimal solution identified by Sethi, 3) generalization of the argument made by Anderson and Salant to continuous time, 4) modification of the argument of Anderson and

[10]We have extended this theoretical exercise to a numerical level as well, by setting up a model (with the same parameter values as the ones that Sanders uses in his trachoma illustration) and solving for the optimal policy for $w = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

Salant by analytically and numerically experimenting with the strategy of pulsing over unequal time intervals.

While the major question that we presented in the first chapter of this thesis still stubbornly persists - namely, why do we get different solutions in discrete and continuous time – we have been successful in clarifying some issues revolving around this puzzle. First, the application of discrete-time dynamic programming to this problem is arguably reasonable. Dynamic programming is a well-suited tool to problems where the decision times can be divided into discrete pieces and the state space has a finite number of values (Lenhart and Workman 2007). Both of these features seem to characterize our problem. Furthermore, the solution that Sanders identifies in discrete time is optimal for discrete-time models with periods of length w. Secondly, the analytical generalization of the argument of Anderson and Salant to continuous time seems to be particularly challenging mathematically. Thirdly, the solution that Sethi identifies in continuous time is correct; this chapter has provided a second approach, besides the Green's theorem approach that Sethi takes to verify the optimality of the singular control. Fourthly, we have been able to numerically find a pulsing control that dominates the singular control in continuous time. This anomalous result might be due to some conceptual error, or to potential limitations of the computational capabilities of Maple (the software used in this example); as of now, we have unable to identify any errors in this example. Lastly, this work has

raised the possibility that the problem might have not been translated correctly from the discrete-time framework to the continuous-time one. This means that both Sethi and Sanders are providing us with distinct solutions because they are not essentially solving an identical problem. To this end, we have raised the issue of the size of b, the upper bound for the treatment rate, in discrete and continuous time. While Sanders seems to explicitly state that b cannot be any greater than one, Sethi does not provide any upper bound for this upper bound. When interpreting his results, he even considers the case when $b \to \infty$ (Sethi 1974, 684). This raises more general questions on how do we treat the inequality constraints on the control in a control-constrained optimization problem when passing from a discrete-time model to a continuous-time one. Section 5 of this chapter provides a possible suggestion on these questions, which is seemingly at odds with Sethi's continuous-time interpretation of b.

Lastly, we would like to remark that the controversy between Sanders and Sethi is not the only, and maybe not even the first, time that economists have encountered different-looking optimal solutions in discrete and continuous time. Spence and Starrett (1975) provide a theorem that establishes an instance when the most rapid approach path is optimal in the continuous-time problem but not in its discrete-time counterpart. Under the time constraints of this thesis, we have not been able to investigate whether the conditions of this theorem hold in the case of the problem of Sanders and

Sethi, but further research might show whether Spence and Starrett's results are applicable or not.

The following chapter turns to a more recent debate that seems to be related to, and at some degree even rooted in, the disagreement between Sethi and Sanders. The resurfacing of this issue is further proof of the persistence of the puzzle discussed in this chapter.

CHAPTER 4

Essay Two: Good questions persist

"I wish I had an answer to that because I'm tired of answering that question."

Yogi Berra

This second essay will focus on a more recent pair of articles – the article by Rowthorn et al. (2009) and the article by Anderson et al. (2011) – the arguments in which are marked by the tension between discrete-time and continuous-time formulations discussed in the previous essay. We will first present the arguments made in each of the articles, highlighting the similarities and differences between these arguments and the classical arguments made by Sanders and Sethi. Then, we will apply the method of switching functions to the continuous-time article by Rowthorn et al. A discussion of the difficulties of obtaining analytical solutions in continuous time will follow. Lastly, we conclude with a summary of the insights obtained by our analysis.

1. Expository discussion

Both the article by Anderson et al. and the article by Rowthorn et al. examine the optimal allocation of a limited per-period budget in minimizing

the discounted social cost of the total infection level in two interconnected subpopulations. So, the dilemma that the social planner faces is how to divide the funds between the two subpopulations, given that these subpopulations start off with different initial infection levels. In general terms, the set of choices available to the social planner can be categorized into two large categories: 1) policies that devote the entire budget to only one subpopulation each period, and 2) policies that treat people from both subpopulations in each period. As it will become clearer later in this discussion, the policies in the first category can be viewed as strictly bang-bang policies, while the policies in the second category can be treated as singular (or intermediate solutions). This broad categorization of admissible policies will shape our discussion of these two articles in the spirit of the classical debate covered in the previous chapter.

Methodologically, these two articles differ in the time scale that they choose in their models: one of the papers considers the problem in discrete time, the other in continuous time. The article by Rowthorn et al., the earliest of the two, considers the continuous-time model, and observes that it is extremely difficult to obtain the optimal solution analytically. In order to circumvent this difficulty, Anderson et al. transferred the model in discrete time and employed dynamic programming to find the optimal discrete-time policy. In order to keep up with the order established in the first essay, we summarize the major points of the discrete-time article first, and then proceed with the findings of the continuous-time one.

The discussion in this essay is limited to the case of two subpopulations, although both articles extend their results further to the general case of more than two subpopulations. Also, we will limit our discussion to the case of per-period budget only, although results can be obtained for the case of a one-time endowment that needs to be managed over time as well. Having set these limitations, we proceed with the article by Anderson et al. first.

The discrete-time minimization problem that Anderson et al. look at is the following:

(4.1)

$$\begin{aligned}
&\underset{F^A,\ F^B}{\text{minimize}} && \sum_{t=1}^{T} \delta^{t-1}(s^A I_t^A + s^B I_t^B) \\
&\text{subject to} && I_{t+1}^i = (1-\mu^i)I_t^i + \frac{\beta^i I_t^i + \chi^i I_t^j}{N^i + N^j}(N^i - I_t^i) - \alpha^i F_t^i \text{ for } i, j = A, B, \\
&\text{with} && p^A F_t^A + p^B F_t^B \le M_t,\ 0 \le F_t^i \le I_t^i,\ \frac{M_t}{p^i} \le I_t^i \le N^i, \\
&\text{and} && I_1^A,\ I_1^B \text{ given.}
\end{aligned}$$

where I^i is the number of infected people for subpopulation i, F_i is the number of treated people in subpopulation i, s^i is the social cost due to an additional infected person in subpopulation i, δ is the discount factor, μ^i is the rate of spontaneous recoveries (lucky recoveries that happen despite treatment) in population i, β^i is the transmission rate in group i, χ^i is the rate at which infected individuals in group j cross-infect people in group i, α^i is the success rate of the treatment, p^i is the cost of treatment per person in population i, M_t is the budget for use in period t, and I_1^i is the initial infection level in group i. While the more interesting case is the one that

allows for interaction between the subpopulations, Anderson et al. start their analysis with the base case $\chi^i = 0$ and establish the concavity of the SIS dynamics and the value function with respect to the infection level. As it turns out, it is very easy to carry these results to the case when $\chi^i \neq 0$.

The objective function in this minimization problem is linear in the state variables I^A and I^B, while the dynamics of the problem are linear in the controls F^A and F^B. The value function corresponding to this problem will be:

$$V_t(I_t^A, I_t^B) = \min_{F_t^A, F_t^B} \{s^A I_t^A + s^B I_t^B + \delta V_{t+1}(I_{t+1}^A, I_{t+1}^B)\}$$

This linearity (or nonconvexity, as Anderson et al. refer to it) in the objective function is problematic to the application of standard dynamic programming techniques in the case when it is possible to treat every infected individual in a certain group and use the left-over budget to treat some infected individuals in the other group. In such a case, in the backward recursion of dynamic programming, every value function will have kinks (so it will not be differentiable everywhere) and it will be only piecewise concave. The inclusion of the assumption of tight budgets (budgets that are not sufficient to treat everyone in a certain subpopulation) reduces the state space. because now we are not considering $0 \leq I_t^i \leq N^i$ but instead we are considering only a subset of it, namely $\frac{M_t}{p^i} \leq I_t^i \leq N^i$ for a group i. So, the reduced state space will now be a rectangle in the (I^A, I^B)-plane, with sides $N^A - I_t^A$ and $N^B - I_t^B$. As Anderson et al. argue, "the standard algorithm would first

establish properties of the cost function and policy rule that hold over the entire state space; only afterward would it use the initial condition and transition rule to determine the optimal trajectory through the subset of that space" (Anderson et al 2011, 4). So, by modifying the standard dynamic programming technique, Anderson et al. establish the strict concavity of the value function with respect to the state and control variables for this subset of the state space first. Then, they use this concavity to make the argument that the optimal policy is always a corner policy.

The recursive nature of the dynamic programming techniques is very helpful when establishing the strict concavity of the value function for every period. In the last period, the value function will be linear in I^i, as the optimal policy is to treat nobody from either group. But, in the second-to-last period, the value function is concave due to the concave dynamics of the SIS disease that come into play through the recursive formulation of V_{T-1} in terms of V_T. Once we have established the strict concavity of V_{T-1}, the value function of every previous period will be a sum of a linear function (the momentary costs) and a strictly concave function (the value function of the next period). The strict concavity of the value function with respect to the infection levels implies its concavity with respect to the treatment levels. Hence, the indifference curves (i.e. "level curves" of equal social discounted cost) are strictly concave in the (F^A, F^B)-plane, while the budget constraint line will have slope $\frac{-p^B}{p^A}$ in this plane. Hence, the optimal policy will be a corner policy that assigns all the available funds to the treatment of a

portion of the infected individuals in one of the subpopulations only. This is the heart of the dynamic-programming argument in Anderson et al. In the next section, we will discuss how Sanders employs a dynamic-programming argument that is very similar to this one.

Therefore, the major result in the work of Anderson et al. is the corner policy that they identify as optimal: when the (tight) budget cannot be transferred from one period to another and there are only two subpopulations isolated from each other, it is always optimal to focus treatment on a single subpopulation only. Then, they show that this result holds in the following three more general cases : a) when there are more than two subpopulations, 2) when the subpopulations interact, provided that $\chi^i < \beta^i$, and 3) when there is a one-time endowment instead of a per-period budget, provided that in no period is there sufficient wealth to treat every infected individual in every subpopulation. Furthermore, in the budget-constrained problem with two or more subpopulations, Anderson et al. show that if the subpopulations share the same infection dynamics, treatment price, and social cost (so they differ in the initial infection level only), the subpopulation that will receive treatment will be the one with the lowest initial level of infection.

Rowthorn et al. considered the continuous-time version of this problem. In their analysis, they do not make use of the assumption of tight budgets, because as we mentioned above, this assumption was employed by Anderson et al. precisely in order to circumvent the difficulty of obtaining analytical

solutions pointed out in the article by Rowthorn et al. Therefore, the upper bound of F^i is I^i. The continuous-time problem is written in the following form:

$$\begin{aligned} &\underset{F^A,\ F^B}{\text{minimize}} && \int_0^\infty e^{-\delta t}(I^A + I^B) \\ &\text{subject to} && \frac{dI^i}{dt} = (\beta I^i + \chi I^j)(N - I^i) - \mu I^i - \alpha F^i \text{ for } i, j = A, B, \\ &\text{with} && c(F^A + F^B) \leq M,\ 0 \leq F^i \leq I^i, \\ &\text{and} && I_0^A,\ I_0^B \text{ given.} \end{aligned} \tag{4.2}$$

Most of the notation is the same as before. The constant c denotes the price of the treatment per person: this price is the same for both subpopulations. If we compare this problem to the discrete-time problem discussed above, we notice several differences. First, as we already noted, the treatment price for each of the subpopulations is equal: $p^A = p^B = c$. Secondly, the marginal social cost for each subpopulation is normalized at $s^A = s^B = 1$. In other words, this problem is concerned with minimizing the total infection in both subpopulations, assuming that an additional infected individual causes equal social harm to his subpopulation in both groups. Thirdly, this continuous-time problem is based on the assumption of identical populations. The parameters β, χ, μ, N, and α that determine aspects of the spread of the infection are equal in both subpopulations.

The optimization method used by Rowthorn et al. is Pontryagin's Maximum Principle. The authors combine analytical and numerical approaches

to evaluate the performance of three strategies under the assumption of limited budgets (i.e. when at least for some intervals of the time horizon, the total number of infected individuals in the two subpopulations exceeds the availability of treatment): a) treating in the region with higher infection level first, b) treating in the region with lower infection level first, c) applying an intermediate level of treatment in both subpopulations simultaneously. A brief description of the analytical argument made by Rowthorn et al. is useful for our further discussion in the third section of this essay.

The case when the available budget is sufficient to treat every infected individual in each of the subpopulations is trivial in terms of minimization of costs, as it is always optimal to treat everybody. Eventually, the infection is either totally eradicated or driven down to some equilibrium in each of the subpopulations. The case when $I^A + I^B \geq \frac{M}{c}$ is the one of interest to us. Rowthorn et al. set the Hamiltonian function, which includes the negative of costs, as well as two co-state variables λ_1 and λ_2 accompanied by the dynamic equation for each of the subpopulations. The relationship between the control variables $F^B = \frac{M}{c} - F^A$ simplifies the Hamiltonian; we could rewrite the Hamiltonian function in terms of just one control, and then attempt to maximize it in terms of that control only, say F_A. The Hamiltonian function is linear in this control:

$$H = -e^{-\delta t}(I^A+I^B)+\lambda_1[(N-I^A)(\beta I^A+\chi I^B)-\mu I^A]+\lambda_2[(N-I^B)(\beta I^B+\chi I^A)-\mu I^B]$$

$$-\lambda_2\alpha\frac{M}{c}+\alpha(\lambda_2-\lambda_1)F^A$$

This linearity in the control makes it very easy to maximize the Hamiltonian when the coefficient before the control is nonzero. As we have seen in chapter 2 and 3, problems arise when this coefficient is zero. By taking time derivatives of this coefficient and setting them equal to zero, we obtain a singular solution for the system. The most rapid approach path (MRAP) will consist of reaching this solution as fast as possible and staying in it for as long as possible.[1] After identifying this singular solution, Rowthorn et al. check its optimality. In order to prove that the most rapid approach path is the less optimal of all possible paths, they replace $-e^{-\delta t}(I^A + I^B)$ with $e^{-\delta t}(I^A + I^B)$ in the Hamiltonian function and show that this function is concave in I^A, I^B, and F^A. The concavity of the Hamiltonian establishes that the Mangasarian's sufficiency conditions for maximality hold for the most rapid approach path, which means that this path maximizes this positive-cost Hamiltonian. Hence, MRAP is the maximizing, rather than the minimizing strategy for this problem. We return to this argument in the third section, where we analyze the optimality character of the singular path.

Rowthorn et al. numerically establish that the optimal path is to treat as many infected individuals as we can in the subpopulation with the lowest

[1] In the finite horizon problem, the optimal solution would be to apply this singular solution for as long as it is possible, before deviating from it in order to meet the endpoint requirement. In the infinite horizon, though, the optimal path is to stay in the singular solution forever.

level of infection first. Only after having treated all the infected individuals in this subpopulation, the planner can use the left-over budget to treat infected individuals from the other subpopulation. So, they claim that the optimal solution for the continuous-time problem is:

$$\text{if } I^i < I^j \text{ then } F_i = \min(I_i, \frac{M}{c}) \text{ and } F_j = \frac{M}{c} - F_i$$

$$\text{if } I^i = I^j \text{ then } F_i = \min(I_i, \frac{M}{c}) \text{ and } F_j = \frac{M}{c} - F_i \text{ or vice versa}$$

It is important to notice that the optimal solution that Rowthorn et al. identify in continuous time is identical to the one identified by Anderson et al. in discrete time, provided that the subpopulations are identical and the budgets are tight.

2. Similarities and differences

This section will provide a comparison of the problem considered by Sanders and Sethi (referred to as problem 1 hereafter in this section) and the problem considered by Rowthorn et al. and Anderson et al. (referred to as problem 2 hereafter in this section). The problems can be compared with respect to: (i) their mathematical formulation, (ii) the economic context captured by the model, (iii) methodology being employed, and (iv) results.

The first and the second aspect –the mathematical formulation and the underlying economic context – are closely related. Problem 1 considers the case of a single population, while problem 2 considers the case of multiple populations, interconnected or not with each other. So, problem 2 has added

a spatial context to the spread of infectious diseases. Also, this additional feature complicates the problem, because the number of state and control variables increases by increasing the number of the subpopulations. Secondly, both problem 1 and problem 2 are concerned with the optimization of funds to control diseases of similar nature. Both problems have similar SIS law of motions, although problem 2 accounts for spontaneuous recoveries from the infection as well. We anticipate that this difference does not change much in the dynamics of the disease.

What is of major importance is the way in which the control enters the law of motion in these two problems. In problem 1, the control is described as a treatment effort: the planner is able to screen everyone, identify the infected individuals in the population, and then treat them with a certain effort level (which takes values between 0 and a maximal treatment level b). In problem 2, on the other hand, the planner chooses the number of infected individuals that will be treated, and each of them gets well with probability α, which is the success rate of the treatment. The control variable in this second problem is F^i, which is also bounded between 0 and $\min\{I^i, \frac{M}{c}\}$. Therefore, the control variable in the first problem appears as: $x' = f(x) + g(x)u$, while in the second problem it appears in the form: $x' = f(x) + ku$ (where x is the state variable, u is the control variable, k is a constant, and the problem is autonomous).

Thirdly, the momentary costs in both problems are of linear form. In problem 1, the costs are linear in both the state variable and the control,

while in the second problem, the costs are control in the state variables only. The control does not appear in the cost. Also, both problems consider the discounted costs over a finite/infinite horizon.

Fourthly, the existence of a budget constraint in the problem is of major importance and economic significance. In problem 1, the planner needs to minimize the costs of the control of an infectious disease without worrying about any per-period budget or endowment constraints. This lack of a budget constraint is compensated by the existence of an upper bound on the treatment effort that can be implemented. In problem 2, the existence of a budget (or one-time endowment) constraint imposes an upper bound on the cumulative number of people that can be treated in *both* subpopulations. Particularly, the assumption of a tight budget imposes an upper bound on the number of the infected people that can be treated in *each* of the subpopulations. Also, it might be useful to note that when the upper bound on F^i is simply I^i, this upper bound is changing over time, unlike in the world of the problem of Sethi and Sanders, where b is fixed over the entire horizon.

There are major similarities between the approaches taken in solving the two problems in discrete and continuous time. Sanders uses dynamic programming in order to solve the discrete-time version of problem 1. As it was thoroughly explained in the previous chapter, he exploits the concavity of the value function with respect to the infection level and the treatment effort in order to establish that the optimal policy is always a strictly bang-bang

policy. The argument of Anderson et al. is very similar in nature: Anderson et al. use precisely this strict concavity to establish that only one subpopulation should be treated in every period. Secondly, we notice that the continuous-time versions of both problems require a discussion of the form of the Hamiltonian function. Both Sethi and Rowthorn et al. make use of the Pontryagin's maximum priniciple. The scalar nature of the state variable in Sethi's problem enables the use of Green's theorem in order to identify the singular solution and confirm its optimality. Such an opportunity does not arise in Rowthorn et al.'s problem, which has two state variables. The switching function method can be employed in both cases: Sethi employs this method in order to find the form of the concatenations of the optimal solution, while Rowthorn et al. make use of the switching function, but do not refer to it explicitly. Relatedly, the Generalized Legendre-Clebsch necessary condition can be checked for both problems as well. The next section will apply this condition to the continuous-time problem 2.

In terms of the results drawn for both problems in the four articles under consideration, the major difference between the solutions obtained for problem 1 and problem 2 is that the discrete-time and the continuous-time optimal solutions agree in the case of the second problem but not in the case of the first problem. Rowthorne et al. and Anderson et al. agree that the optimal solution of problem 2 is to spend all the available budget on treating one subpopulation only. Meanwhile, Sethi and Sanders don't agree on whether a strictly bang-bang solution is optimal. Nevertheless, in

both these problems, we see the discussion of the same type of solutions –namely, bang-bang and singular solutions. An interesting question is the following: can we draw an analogy between what a bang-bang policy and a MRAP is in Sanders' and Sethi's world and what it is in Anderson et al.'s and Rowthorne et al.'s world? In problem 1, a bang-bang policy is that you either treat as intensively as you can or you don't treat anyone in the single population under consideration. Similarly, in problem 2, a bang-bang policy consists in the maximal treatment of the infected individuals in one subpopulation and the total lack of treatment for the other subpopulations (under the assumption of tight budgets). In both problems, a singular control signifies an intermediate level of treatment: in the case of only one subpopulation, an intermediate level of treatment is between 0 and b, while in the case of multiple subpopulations, an intermediate level of treatment is between 0 and $\min\{I^i, \frac{M}{c}\}$ for each of the subpopulations (so funds are divided between all subpopulations). This similarity in the solutions that appear in these two problems reinforces the fundamental similarity between these two optimization problems and the persistence of essentially the same issues for this class of problems.

3. Switching-function analysis of the continuous-time case

Continuing the switching function analysis started in the previous chapter, this section will discuss the optimality of the singular solution identified

by Rowthorn et al. The aim of this section is threefold: 1) to reinterpret Rowthorn et al.'s procedure of identifying a singular solution in terms of the sign of the switching function, 2) to check whether the Generalized Legendre-Clebsch necessary condition for maximality holds, and 3) to reevaluate Rowthorn et al.'s argument that identifies MRAP as the least optimal solution.

The Hamiltonian function for the problem of Rowthorn et al. is the following:

$$H = -e^{-\delta t}(I^A + I^B) + \lambda_1[(N - I^A)(\beta I^A + \chi I^B) - \mu I^A - \alpha F^A] +$$

$$+\lambda_2[(N - I^B)(\beta I^B + \chi I^A) - \mu I^B - \alpha F^B].$$

where λ_1 and λ_2 are the costate variables. Knowing that all funds that will not be spent in one subpopulation will be spent in the other, i.e. $F^B = \frac{M}{c} - F^A$, we rewrite the Hamiltonian function in the following form:

$$H = -e^{-\delta t}(I^A+I^B)+\lambda_1[(N-I^A)(\beta I^A+\chi I^B)-\mu I^A]+\lambda_2[(N-I^B)(\beta I^B+\chi I^A)-\mu I^B]$$

$$-\lambda_2\alpha\frac{M}{c} + \alpha(\lambda_2 - \lambda_1)F^A$$

So, now the Hamiltonian contains only one control variable, F^A. As the Hamiltonian is linear in the control, we can identify the switching function:

$$W = \alpha(\lambda_2 - \lambda_1).$$

When $W > 0$, so when $\lambda_2 - \lambda_1 > 0$, we need to make F^A as large as possible in order to maximize the Hamiltonian. The opposite holds for the case when

$W < 0$ ($\Leftrightarrow \lambda_2 - \lambda_1 < 0$). The problematic case is when $W = 0 \Leftrightarrow \lambda_2 - \lambda_1 = 0$. In this case,

$$\lambda_2 = \lambda_1. \tag{4.3}$$

Then, we take the first time derivative of W and set it equal to zero:

$$W' = 0 \Leftrightarrow \lambda_2' - \lambda_1' = 0$$

From the canonical equations of the maximum principle, we know that:

$$\lambda_1' = -\frac{dH}{dI^A} =$$

$$= -e^{-\delta t} + \lambda_2[N\chi - \chi I^B] + \lambda_1[N\beta - 2\beta I^A - \chi I^B - \mu]$$

$$\lambda_2' = -\frac{dH}{dI^B} =$$

$$= -e^{-\delta t} + \lambda_1[N\chi - \chi I^A] + \lambda_2[N\beta - 2\beta I^B - \chi I^A - \mu]$$

Therefore, combining equation (4.3) with these two last equations, we get the following:

$$\lambda_1' = \lambda_2' \Leftrightarrow I^A = I^B \tag{4.4}$$

$$W' = I^A - I^B$$

The first time derivative has given us the singular state path, but not an expression for the singular control. We need to take the second time derivative in order to determine the singular control:

$$W'' = (I^A)' - (I^B)' = 0$$

After several algebraic transformations, and after taking into account equation (4.3) and (4.4), we get that:

$$W'' = -\alpha F^A + \alpha F^B = \alpha(\frac{M}{c} - F^A - F^A) = \alpha\frac{M}{c} - 2\alpha F^A$$

The singular control is:

$$F^A = \frac{M}{2c} = F^B \tag{4.5}$$

In order to check whether the Generalized Legendre-Clebsch condition holds, we notice that the singular control that we have found is of first order. So,

$$(-1)^1\frac{d}{dF^A}W'' = (-1)(-2\alpha) = 2\alpha \geq 0$$

Therefore, the Legendre-Clebsch condition for maximality does not hold. The singular control $F^A = \frac{M}{2c}$ is not optimal, as it does not satisfy this necessary condition for optimality.

After having presented the switching function analysis and having verified that the singular solution is not optimal, we return to a brief discussion of the method that Rowthorn et al. use in order to argue for the non-optimality of the MRAP. In Appendix A.3. of the article (Rowthorn et al 2009, 8), the authors employ the Mangasarian's sufficiency conditions for maximality in order to show that MRAP maximizes, rather than minimizes the total costs. They consider the Hamiltonian of the positive momentary costs:

$$H = e^{-\delta t}(I^A+I^B)+\lambda_1[(N-I^A)(\beta I^A+\chi I^B)-\mu I^A]+\lambda_2[(N-I^B)(\beta I^B+\chi I^A)-\mu I^B]$$

$$-\lambda_2\alpha\frac{M}{c}+\alpha(\lambda_2-\lambda_1)F^A$$

and show that if this Hamiltonian function is strictly concave in I^A, I^B, and F^A in a neighborhood around the singular control, then the singular control maximizes the positive costs. In this exercise, I will extend the discussion a bit further in order to show why this argument does not work for the original Hamiltonian function, i.e. why cannot we establish the concavity of the original negative-momentary-cost-based Hamiltonian in this same fashion. The Mangasarian's sufficient condition of maximality requires the following Hessian matrix to be negative semi-definite:

$$M=\begin{bmatrix} \frac{d^2H}{(dI^A)^2} & \frac{d^2H}{dI^BdI^A} & \frac{d^2H}{dF^AdI^A} \\ \frac{d^2H}{dI^AdI^B} & \frac{d^2H}{(dI^B)^2} & \frac{d^2H}{dF^AdI^B} \\ \frac{d^2H}{dI^AdF^A} & \frac{d^2H}{dI^BdF^A} & \frac{d^2H}{(dF^A)^2} \end{bmatrix}$$

After finding these derivatives:

$$M=\begin{bmatrix} -2\beta\lambda_1 & -\chi(\lambda_1+\lambda_2) & 0 \\ -\chi(\lambda_1+\lambda_2) & -2\beta\lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

M will be negative semi-definite if: i) $\lambda_1\geq 0$, and ii) the determinant of the matrix is nonnegative, i.e. $4\beta^2\lambda_1\lambda_2-\chi^2(\lambda_1+\lambda_2)^2\geq 0$.

Up to this point of the analysis it is normal to wonder why is this analysis any different for the original $(-e^{-\delta t}(I^A+I^B))$-based Hamiltonian function, given that all the terms in the matrix are second derivatives of this function, which, due to the linearity of the cost function, are not affected at all by the

sign of the cost function. Also, for $\lambda_1 = \lambda_2$ along the singular control path, we notice that condition (ii) holds, provided that $\beta > \chi$.[2] The condition that is dependent on the sign of the costs is condition (i). The costate variable measures the marginal value of an additional infected person in the population; condition (ii) requires an additional infected individual to bring positive marginal benefits to the population. This clearly holds only if we are aiming to maximize the number of infected people, not to minimize it (which is the case when we aim to maximize the Hamiltonian containing the negative momentary costs). When trying to minimize costs, the costate variables are in fact nonpositive, following the same reasoning. So, this first condition is the reason why we cannot establish the concavity of the original Hamiltonian, which would give us the optimality of the singular solution.

The two exercises of this section have analyzed further why the singular solution is not optimal for the continuous-time problem of Rowthorn et al.

4. Conclusions

This second essay has briefly reviewed two recent articles on the allocation of limited funds to control the spread of an SIS infectious disease throughout multiple subpopulations. The essay has argued that the problem considered in this pair of articles is very similar to the optimization

[2]It is sensible to assume that $\beta > \chi$, which, in other words, states that the likelihood that an infected person infects a susceptible person from his group is higher than the likelihood that he infects a susceptible person from the other group, for instance, due to the fact that the groups are geographically distant from each other.

problem treated by Sanders and Sethi. Furthermore, this pair of articles is characterized by a similar-looking tension between the discrete-time and the continuous-time framework; what is different in this tension, though, is that the tension is not due to the disagreement of the discrete-time and continuous-time optimal solutions, but to the methodology used to reach these solutions. Obtaining analytical solutions in the discrete-time framework becomes possible only due to the additional "tight budget" assumption that Anderson et al. include in their analysis. On the other hand, an analytical solution for the continuous-time problem is hard to find; the continuous-time optimal solution is identified by numerical means only. Lastly, this essay has extended the discussion of the non-optimality of the singular solution by employing the Generalized Legendre-Clebsch condition: this extension has provided us with an optimization problem that looks very similar to the problem of Sethi and Sanders, but in which this necessary condition fails.

To conclude, it is important to emphasize the fact that questions concerning the superiority of bang-bang solutions over singular solutions and *vice versa* continue to persist for this class of SIS optimization problems. There is, apparently, something special in the structure of this problem that creates tensions between these two types of optimal solutions. Had we had more time to work on this recent pair of articles, we would have liked to extend our discussion to the applicability of tools from continuous dynamic

programming. More specifically, we would have researched on possible modifications of the standard continuous dynamic programming methods that are analogous to the modified dynamic programming argument made by Anderson et al. This would have most likely required for the inclusion of the "tight" budget assumption in continuous time, which is not assumed in the work of Rowthorn et al. Another potentially fruitful path would have been to analyze the sensitivity of the Maximum Principle-based method explored in Rowthorn et al. to the introduction of the "tight" budget assumption.

CHAPTER 5

Concluding remarks

"Funeral after funeral, theory advances."

Paul A. Samuelson

This final chapter will briefly summarize some of the results drawn in the previous two essays, discuss potential implications for public health policy-makers, and review the old and new puzzles that have persisted throughout or arisen along the analysis in this thesis.

1. Major results revisited

Our interest in the discrepancies between discrete-time and continuous-time optimal solutions and optimization techniques originally arose out of the recently published article by Anderson et al. This article seemed to offer a way to bypass the difficulty of obtaining analytical continuous-time solutions by a simple modification of the standard dynamic programming technique in discrete time. The article naturally incited questions such as: Can we identify any virtues characteristic of the discrete-time problem – not possessed by the continuous-time problem – that enabled scholars to obtain analytical solutions in discrete time but not in continuous time? Is there any way to extend the argument of Anderson et al. in continuous

time? If not, what is so problematic about continuous-time formulations? As we investigated the literature on the topic, we noticed that a similar kind of discrepancy between discrete and continuous time – albeit related to the form of the optimal solution rather than to methodological difficulties of obtaining analytical solutions at all – had marked this literature since its beginnings, with the articles by Sanders and Sethi. This observation enriched our research in ways that were unimaginable to us at the start of this thesis project.

Issues with formulations of SIS control models in discrete and continuous time arise out of the particular form of these optimization problems: typically, the cost function is linear in the infection level and the treatment control variable, and the law of motion of an SIS disease is nonlinear. In the first essay, we evaluated the arguments offered by Sanders and Sethi, and we concluded that as we cannot find any faults with their arguments – in fact, we were able to check the optimality of the singular solution in continuous time by an alternative approach as well, – we are bound to fail to reject the hypothesis that the discrete-time and continuous-time optimal solutions should be similar-looking. The time scale chosen in a model might fundamentally alter the shape of the optimal solutions in discrete and continuous time. Secondly, in this essay, we have provided a numerical example that examines the dominance of the singular control proposed by Sethi by

a pulsing strategy. Given that we have analytically established the optimality of the singular control, such a numerical example strikes us as highly anomalous.

The second essay returned the attention to our original interest on the articles by Rowthorn et al. and Anderson et al. Although using a slightly different specification in their model, these articles confirm the tension between singular and bang-bang controls and between dynamic programming and Pontryagin's maximum principle previously observed in the work of Sanders and Sethi. By a comparative analysis, we noticed the similarities between the two pairs of articles in terms of the model specifications, the techniques being used, and the form of the optimal solutions. In the articles of Rowthorn et al. and Anderson et al., though, the singular control is not optimal in continuous time. This shows the high sensitivity of the optimality of the singular control on the specification of the model: although Sethi's and Rowthorn et al.'s models are very similar, the singular control is optimal in one but not in the other. We confirmed the nonoptimality of the singular control in this case by employing the Generalized Legendre-Clebsch necessary condition as well.

2. The conundrum of the "time scale effect" and public health policies

The central problem of this thesis revolves around a normative modelling issue: namely, how should we treat the flow of time in modelling the

control of SIS infectious diseases? There are three parts to this dilemma: 1) an infection spreads in continuous time, 2) continuity of time implies continuity of the state variable (the number of the infected people), while in fact people get sick in a discrete fashion, and 3) decisions are made in discrete time, usually in monthly or annual intervals. The question, then, of how frequently we can make decisions depends highly on the type of the SIS disease being fought, the type of treatment being applied, the technology in screening and identifying infected individuals etc. The horizon chosen for the model – either finite or infinite – depends on whether we are modelling a seasonal infection or a permanent one. In a finite horizon problem, the factors that determine the endpoint constraint, the target infection level, are crucial. Also, in the real world, the limitations of our treatment capabilities are always changing due to rapid medical advances. Budgets vary, subpopulations merge, people die. All these issues need to be kept in mind when attempting to draw implications of these models to the public health policies intended to fight infectious diseases.

Admitting the real-world limitations, these models nevertheless offer a general framework to guide our thinking of how to optimally control infectious diseases under limited funds and limited medical capabilities. The strictly bang-bang solutions of Sanders suggest that treating below your maximal capabilities – at an intermediate level of treatment – is never optimal: you either treat everyone that you can, or you don't treat anyone. The singular solution of Sethi, on the other hand, aims to keep a stable

number of infected people in the population. When applied to the real world, though, "staying in the turnpike infection level" is not an easy task to do: random disturbances will constantly drive the number of infected people away from the turnpike, and the public health officials might need to constantly apply bang-bang policies to get back to the turnpike. The articles by Rowthorn et al. and Anderson et al., while being more unanimous on their recommendations for public health policies, suggest that the region with the lowest level of infection should deserve all the funds first. This is very counterintuitive because of two widely-spread misperceptions: a) funds should be divided among populations, b) funds should go to the areas where the infection levels are more dramatic first. So, these policy prescriptions, while still debated and grounded in a quite abstract framework, provide food for thought for policy-makers when dealing with optimal ways to control SIS-driven diseases.

3. Puzzles: Old and new

We conclude this final chapter with a list of puzzles, old and new, that have been present or have emerged while analyzing the discrepancies between discrete and continuous time optimal solutions. While we have answered some of our initial questions (under the time constraints and the mathematical limitations present), many puzzles remain unsolved. The following questions attempt to capture and formulate some of these puzzles:

- Can we provide a mathematically rigorous argument that shows *why* the singular solution is not optimal in discrete time in Sanders' model, while it is in the continuous-time model in Sethi's article?
- In the framework of a single population, would the continuous-time solution analogous to Sanders' solution look more like pulsing or chattering? In other words, as the time discretization approaches zero, does the number of switchings go off to infinity, or does it remain finite?
- What could explain the anomalous numerical result that we have presented in the third chapter?
- The behavior of the upper bound on the treatment effort in Sanders' article as the time discretization becomes finer remains not fully explored.
- Are there any other alternative ways, different from the ones we have already pursued, to extend the argument made by Anderson and Salant in continuous time?
- Are there any hidden factors that are making the solution of the singular control optimal in Sethi's problem but not in Rowthorn et al.'s problem?
- How applicable is the discrete maximum principle to Sethi's problem? How useful (if at all) is continuous dynamic programming in extending the modified dynamic programming argument of Anderson et al. in continuous time?

Appendices

APPENDIX A

Starting from an infection level x_{t+1}, we can find γ_{t+1} that would restore the infection level at x^s in the next period. The difference equation for the next time period is:

$$x^s = x_{t+1} + w\big(\beta x_{t+1}(N - x_{t+1}) - \gamma_{t+1}x_{t+1}\big)$$

Substituting for x_{t+1}, we can solve for γ_{t+1}:

$$x^s = x^s(1-wh) + w\bigg(\beta x^s(1-wh)\big(N - x^s(1-wh)\big) - \gamma_{t+1}x^s(1-wh)\bigg)$$

$$\Downarrow$$

$$\gamma_{t+1} = \frac{h}{hw-1} + \beta\big(N - x^s(1-wh)\big) \tag{5.1}$$

Now that we have expressions for x^s, γ^s, x_{t+1}, and γ_{t+1}, we can calculate the costs for these two periods:

$$\begin{aligned}
&\delta^t\Big[wCx^s + wK(\gamma^s + h)\Big] + \delta^{t+1}\Big[wCx_{t+1} + wK\gamma_{t+1}\Big] \\
&= \delta^t\Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)} + wK\big(\beta(N-x^s) + h\big)\Big] + \\
&+\delta^{t+1}\Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)}(1-wh) + wK\bigg(\frac{h}{hw-1} + \beta\big(N - x^s(1-wh)\big)\bigg)\Big]
\end{aligned} \tag{5.2}$$

Dividing through by δ^t and simplifying the expression, we get:

$$\Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)}+wK\big(\beta(N-x^s)+h\big)\Big]+$$
$$\delta\Big[wC\frac{(1-\delta)K}{w\delta(C-K\beta)}(1-wh)+wK\Big(\frac{h}{hw-1}+\beta\big(N-x^s(1-wh)\big)\Big)\Big]$$
$$=\Big[C\frac{(1-\delta)K}{\delta(C-K\beta)}+wK\big(\beta(N-x^s)+h\big)\Big]+$$
$$+[C\frac{(1-\delta)K}{(C-K\beta)}(1-wh)+wK\Big(\frac{1}{hw-1}+\beta\big(N-x^s(1-wh)\big)\Big)\Big]$$

(5.3)

For a fixed value of w, we could rearrange the terms by isolating all the terms that include h in them and labeling all the other terms as constants. Therefore, the two-period cost function is:

$$constants+wKh-whC\frac{(1-\delta)K}{(C-K\beta)}+\frac{wK}{hw-1}+w^2K\beta x^s h$$
$$=constants+wK\big(h-C\frac{(1-\delta)h}{C-K\beta}+\frac{h}{hw-1}+\beta wx^s h\big)$$
$$=constants+wK\Big[h\big(1-C\frac{1-\delta}{C-K\beta}+\beta wx^s\big)+\frac{h}{hw-1}\Big] \tag{5.4}$$

Computing the limits:

$$\lim_{w\to 0}x^s=\lim_{w\to 0}(\frac{(1-\delta)K}{w\delta(C-K\beta)})=\frac{K\alpha}{C-K\beta}$$

$$\lim_{w\to 0}\gamma^s=\lim_{w\to 0}\beta(N-x^s)=\beta(N-\frac{K\alpha}{C-K\beta})$$

$$\lim_{w\to 0}x_{t+1}=\lim_{w\to 0}x^s(1-wh)=\lim_{w\to 0}x^s-\frac{Kh}{C-K\beta}\lim_{w\to 0}\frac{1-\delta}{\delta}=$$
$$=\frac{K\alpha}{C-K\beta}-\frac{Kh}{C-K\beta}\lim_{w\to 0}rw=\frac{K\alpha}{C-K\beta}$$

$$\lim_{w\to 0}\gamma_{t+1}=\lim_{w\to 0}(\frac{h}{hw-1}+\beta\big(N-x^s(1-wh)\big)=\lim_{w\to 0}\frac{h}{hw-1}+\beta N-\beta\lim_{w\to 0}(x^s(1-wh))$$

$$= -h + \beta N - \frac{K\alpha\beta}{C - K\beta}$$

APPENDIX B

The closed-form solutions for the differential equations mentioned above are:

$$x_1(t) = \frac{\beta N - \gamma^s - h}{\beta + C_1\beta N e^{-t(\beta N - \gamma^s - h)} - C_1(\gamma^s + h)e^{-t(\beta N - \gamma^s - h)}}$$

for (3.11), and:

$$x_2(t) = \frac{\beta N - \gamma *}{\beta + C_2\beta N e^{-t(\beta N - \gamma_{t+1})} - C_2\gamma * e^{-t(\beta N - \gamma *)}}$$

for (3.12). The constants C_1 and C_2 depend on the initial conditions of the differential equation, so they depend on x^s and $x(t_2)$ respectively. In order to make this point clear, we can start with t_1, at which $x(t_1) = x_0 = x^s$. Then,

$$x^s = \frac{\beta N - \gamma^s - h}{\beta + C_1(\beta N - \gamma^s - h)} \qquad \Rightarrow \qquad C_1 = \frac{\beta N - \gamma^s - h - x^s\beta}{x^s(\beta N - \gamma^s - h)}$$

Our goal at this point is to find the policy γ_{t+1} that we need to apply to return back to the infection level x^s at t_3. The value of γ_{t+1} depends on $x(t_2)$, therefore we need to solve for $x(t_2)$ first, *i.e.* we need to find the new infection level that has been caused by the perturbation in the policy at time t_1. As $t_2 - t_1 = \Delta t$, then:

$$x(t_2) = \frac{\beta N - \gamma^s - h}{\beta + C_1\beta N e^{-\Delta t(\beta N - \gamma^s - h)} - C_1(\gamma^s + h)e^{-\Delta t(\beta N - \gamma^s - h)}}$$

where C_1 is as determined above. Therefore, for the interval $[t_2, t_3]$, the initial level of infection is $x_0 = x(t_2)$. In order to determine the value

of the constant C_2 that appears in the closed-form solution of the second differential equation, we solve for C_2 in the following equation:

$$x_0 = x(t_2)$$

$$\Downarrow$$

$$\frac{\beta N - \gamma_{t+1}}{\beta + C_2\beta N - C_2\gamma_{t+1}} = \frac{\beta N - \gamma^s - h}{\beta + C_1\beta N e^{-\Delta t(\beta N - \gamma^s - h)} - C_1(\gamma^s + h)e^{-\Delta t(\beta N - \gamma^s - h)}}$$

Letting the right-hand side be denoted by $x(t_2)$, the value for C_2 is:

$$C_2 = \frac{\beta N - \gamma_{t+1} - \beta x(t_2)}{x(t_2)(\beta N - \gamma_{t+1}}$$

Now that we have an expression for C_2, we can solve for γ_{t+1}. We know that:

$$x(t_3) = x^s$$

$$\Downarrow$$

$$\frac{\beta N - \gamma_{t+1}}{\beta + C_2\beta N e^{-\Delta t(\beta N - \gamma_{t+1})} - C_2\gamma_{t+1}e^{-\Delta t(\beta N - \gamma_{t+1})}} = x^s$$

$$\Downarrow$$

$$\beta N - x^s\beta = \gamma_{t+1} + x^s\big(e^{-\Delta t(\beta N - \gamma_{t+1})}(C_2\beta N - C_2\gamma_{t+1})\big)$$

It is mathematically intractable to isolate γ_{t+1} from this equation, as γ_{t+1} appears in the exponential terms and in the linear terms of the equation. Therefore, taking the limit of both sides as $\Delta t \to 0$ might be helpful in this case.

$$\lim_{\Delta t \to 0}(\beta(N - x^s)) = \lim_{\Delta t \to 0}\big(\gamma_{t+1} + x^s\big(e^{-\Delta t(\beta N - \gamma_{t+1})}(C_2\beta N - C_2\gamma_{t+1})\big)\big)$$

$$\Downarrow$$

$$\beta(N - x^s) = \gamma_{t+1} + x^s(C_2\beta N - C_2\gamma_{t+1})$$

$$\Downarrow$$

$$\gamma_{t+1} = \frac{\beta N - x^s\beta - x^s C_2\beta N}{1 - C_2 x^s}$$

Nevertheless, this shortcut might cause trouble if we would like to calculate the costs of treatment for time intervals that are not infinitesimally small. We will refer to this point later.

At this point, we have determined the treatment level that need to be applied at t_2 in order to return to the stationary level of infection x^s. The cost function for the interval $[t_1, t_3]$ can be written as a sum of costs of the intervals $[t_1, t_2]$ and $[t_2, t_3]$ (which are not equal to each-other because their respective functions of the infection level and the treatment levels are not equal). Hence, the total costs for $[t_1, t_3]$ are:

$$\int_{\Delta t}^{0} \delta^{t_1+\Delta t}\big(Cx_1(t) + K(\gamma^s + h)\big) + \int_{\Delta t}^{0} \delta^{t_1+\Delta t+\Delta t}\big(Cx_2(t) + K(\gamma_{t+1})\big)$$

and we are interested to evaluate it when $\Delta t \to 0$.

APPENDIX C

During the first time interval $[0, t_1]$, we will apply a treatment level of $\gamma = 0$, therefore the infection level $x_1(t)$ (the infection level during the first time interval) is governed by the differential equation

$$x_1' = 0.2x_1(100 - x_1)$$

Solving this differential equation for an initial level of infection $x_1(0) = x^s$, we obtain the solution:

$$x_1(t) = \frac{700}{7 + 393e^{-20t}}$$

We fix $t_1 = 0.00001$. Therefore,

$$x_1(0.00001) = \frac{700}{7 + 393e^{-20 \cdot 0.1}} = 1.750343908185993587762004647987773229 3$$

$$943346792536407357132106155840412442942117900828453914779269 28...$$

This is the infection level by the end of the interval $[0, t_1]$. At time t_1, we switch to $\gamma = b = 25$. Hence, the differential equation that governs the infection level $x_2(t)$ over the time interval $[t_1, t_2]$ is:

$$x_2' = 0.2x_2(100 - x_2) - 25x_2$$

with initial value $x_2(0) = x(t_1)$. Its closed-form solution is:

$$x_2(t) = \frac{5.469824713081229961756264524961791341857295872667 5 \cdot 10^{49}}{-2.187929885232491984702505809984716536742918349067 \cdot 10^{48} + 3.34379298852324919847025058099847165367429183490 67cdot10^{49}e^{5t}}$$

We will apply $\gamma = 25$ long enough for the infection level to return to the steady-state level $x^s = 1.75$. Therefore, we can solve for the length of the interval over which we need to apply the maximum γ in order to return to that level of infection; we need to solve the equation:

$$x_2(t) = 1.75$$

$$\frac{5.469824713081229961756264524961791341857295872667 5 \cdot 10^{49}}{-2.187929885232491984702505809984716536742918349067 \cdot 10^{48} + 3.34379298852324919847025058099847165367429183490 67cdot10^{49}e^{5t}} = 1.75$$

So,

$t = 0.000036728638910417004916381023817146905453400827670181369261471146746170918659510169099227972718277069 43$

This solution of the equation tells us that the length of the second time interval is

$t_2 - t_1 = 0.000036728638910417004916381023817146905453400827670181369261471146746170918659510169099227972718277069 43$

Hence, the second time interval is

$[0.00001, 0.00001 + 0.000036728638910417004916381023817146905453400827670181369261471146746170918659510169099227972718277069 43]$.

At this point, we can calculate the costs over the two time intervals.

$$\int_0^{0.00001} e^{-0.9t} \left(2.5 \cdot \frac{700}{7 + 393e^{-20t}} \right) dt =$$

$= 0.000043754101813843881970728000215401512943527651372755292536144627261511518594653500509746990401361928 86$

$$\int_0^{0.000036728638910417004916381023817146905453400827670181369261471146746170918659510169099227972718277069 43} e^{-0.9(0.00001+t}$$

$$\left(2.5 \cdot \frac{5.469824713081229961756264524961791341857295872667 5 \cdot 10^{49}}{-2.187929885232491984702505809984716536742918349067 \cdot 10^{48} + 3.34379298852324919847025058099847165367429183490 67cdot10^{49}e^{5t}} + 2\right.$$

$$= 0.00337437334691758743310863609378579721515939218458816526723678958226377357267064343742915075080031415 4$$

Therefore, the sum of costs for both intervals is:

0.00341812744873143131507936409400119872810291983596092055977293420952528509126529693793889774120167608 3

We need to compare this total cost to the cost of applying $\gamma^s = 19.65$ over both intervals (the solution of the respective differential equation $x' = 0.2x(100 - x) - 19.65x$ is the steady-state solution $x^s(t) = 1.75$):

$$\int_0^{0.00001+0.000036728638910417004916381023817146905453400827670181369261471146746170918659510169099227972718277069 43} e^{-0.9t} \left(2.5 \cdot 1.75 + 19.65 \cdot 3.5\right) dt =$$

$$= \mathbf{0.00341812805975036626982488025141824528304698467219420577039191548788758828475135111186964698156184334 5}$$

The difference between the cost of applying $\gamma^s = 19.65$ and the cost of alternating between $\gamma = 0$ and $\gamma = 25$ is:

$$\mathbf{6.11 \cdot 10^{-10}}$$

Appendix D

In this part, we briefly introduce two alternative approaches that we have explored in order to make sense of the numerical result discussed in 3.2. The first one attempts to formalize an intuitive idea about chattering between $\gamma = 0$ and $\gamma = b$ at an instant. Because of the fact that the instantaneous costs of applying a certain policy γ are linear in γ, the comparison of costs between the case of applying γ^s and the case of pulsing between $\gamma = 0, b$ can be reduced in a mere comparison of treatment policies (weighted appropriately based on the share of an instant during which they get applied). Therefore, we need to compare γ^s with $\lim_{t_1 \to 0} \frac{\Delta t}{t_1 + \Delta t} b$, or alternatively, we need to compare $\lim_{t_1 \to 0}(1 + \frac{t_1}{\Delta t})$ with $\frac{b}{\gamma^s}$. We have an expression for Δt as a function of t_1:

$$\Delta t = -\frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_1(\beta N - b)}\right)}{\beta N - b} \tag{5.5}$$

Therefore, in evaluating the limit, we can apply L'Hopital's rule with respect to t_1. After several mathematical transformations, we obtain that

$$\lim_{t_1 \to 0}(1 + \frac{t_1}{\Delta t}) = \frac{b}{\gamma^s} \tag{5.6}$$

This equality suggests that a chattering policy and an intermediate level policy are equally costly in this framework.

The second analytical approach that we have taken deals with the ratio of the costs of Sethi's optimal policy and the chattering policy (in the rest of this discussion, we will refer to the pulsing strategy between $\gamma = 0$ and

$\gamma = b$ as $Sanders1$ and $Sanders2$, as this pulsing strategy is motivated by the optimal solution that Sanders provides). As mentioned above, the overall costs for the pulsing strategy are:

(5.7)
$$\int_0^{t_1} e^{-\alpha t}(C\frac{\beta N}{\beta + C_1\beta N e^{-\beta N t}}) + \int_0^{-\frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_1(\beta N - b)}\right)}{\beta N - b}} e^{-\alpha(t_1+t)}(C\frac{\beta N - b}{\beta + C_2 e^{-(\beta N - b)t}(\beta N - b)} + Kb)$$

So, we are interested in the following limit:

(5.8)
$$\lim_{t_1 \to 0}\left(\frac{Sethi}{Sanders_1 + Sanders_2}\right)$$

Hence,

(5.9)
$$\lim_{t_1 \to 0}\left(\frac{\int_0^{t_1 - \frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_2(\beta N - b)}\right)}{\beta N - b}} e^{-\alpha t}(Cx^s + K\gamma^s)}{\int_0^{t_1} e^{-\alpha t}(C\frac{\beta N}{\beta + C_1\beta N e^{-\beta N t}}) + \int_0^{-\frac{\log\left(\frac{\beta N - \beta x^s - b}{x^s C_2(\beta N - b)}\right)}{\beta N - b}} e^{-\alpha(t_1+t)}(C\frac{\beta N - b}{\beta + C_2 e^{-(\beta N - b)t}(\beta N - b)} + Kb)}\right)$$

Because all these integrals approach 0 as $t_1 \to 0$, we apply L'Hospital's Rule in taking the limit, so:

(5.10)
$$\lim_{t_1 \to 0}\left(\frac{Sethi}{Sanders_1 + Sanders_2}\right) = \lim_{t_1 \to 0}\left(\frac{\frac{\delta}{\delta t_1}Sethi}{\frac{\delta}{\delta t_1}Sanders_1 + \frac{\delta}{\delta t_1}Sanders_2}\right)$$

After several mathematical steps (see below for greater details), we obtain the following expression for the limit of the ratio of costs:

(5.11)
$$\lim_{t_1 \to 0}\left(\frac{\frac{\delta}{\delta t_1}Sethi}{\frac{\delta}{\delta t_1}Sanders_1 + \frac{\delta}{\delta t_1}Sanders_2}\right) = \frac{b(Cx^s + K\gamma^s)}{(b - \gamma^s)Cx^s - \frac{(C\beta N x^s - Kb\beta N + Kb^2)b}{\beta N - b}}$$

We are trying to figure out whether we can specify combinations of values of the parameters for which this expression is greater than 1, *i.e.* a combination of parameter values for which Sethi's strategy can be dominated by chattering. For instance, notice that if we can find a combination of values such that $0 < C\gamma^s x^s + \frac{(C\beta N x^s - Kb\beta N + Kb^2)b}{\beta N - b} < Cx^s b$ (keeping in mind that $x^s = \frac{K\alpha}{C-K\beta}$ and $\gamma^s = \beta(N - \frac{K\alpha}{C-K\beta})$, then it is certain that this limit is greater than 1. The numerical simulations that we were able to run have not yielded any useful results, but we have consistently obtained values that are less than 1 for the limit expression.

Finally, notice that these additional analytical approaches are somehow contradicting each other. The first approach establishes that a chattering policy is as costly as a turnpike policy, while the second one leaves open the possibility that these policies are not equally costly (numerical simulations have shown so far that the turnpike policy dominates the chattering policy).

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