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F6-007-S-Functions And Derivatives In SIR Models

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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

Functions and their Derivatives in SIR Models

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STATEMENT

Introduction to SIR Models

Epidemic spread can be modeled by a system of differential equations. Such models include the SIR (Susceptible-Infectious-Removed) model and its relatives. An example appears in Figure 1.

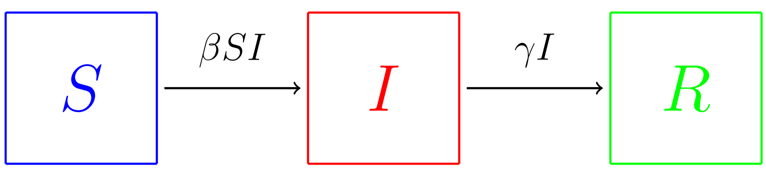


Figure 1. Sample SIR (Susceptible-Infectious-Removed) diagram

In the example of Figure 1, the compartment S consists of Susceptible people, that is, those who could become infected. Compartment I is people who are currently infected and who are able to spread the disease. Once people can no longer spread the disease, whether due to immunity or for other reasons, they move to the Removed compartment, labeled R.

The differential equations that match Figure 1 are

$$\begin{aligned}
\frac{dS}{dt} &= -\beta S I \\
\frac{dI}{dt} &= \beta S I - \gamma I \\
\frac{dR}{dt} &= \gamma I.
\end{aligned}$$

The following simple rules permit the creation of many related models.

1. Each compartment has its own differential equation representing people entering and/or leaving the compartment. For instance, $dS/dt = -\beta SI$ is the equation for the S compartment. Entering and leaving are represented by arrows in the diagram.
2. Positive terms on the right-hand side of each differential equation represent people entering the compartment. Negative terms represent people leaving the compartment.
3. At any given moment, a person must be in exactly one compartment.
4. When all movement is between compartments—that is, no new people enter or leave due to birth, death, immigration, or emigration—the total number of people in the population remains constant, that is, the population size is conserved.

Different outbreaks have different characteristics. There may be incubation periods, or people entering or leaving the population, or isolation of infected individuals, or many other variations. These can be incorporated into models by including different compartments, different arrows into and out of compartments, and different functions on those arrows. To decide how to write a model, modelers consider the outbreak at hand as well as the specific questions they wish to ask about those models.

Determining Outcomes: Visualizing DEs and Their Solutions

A common goal for modeling is to understand likely outcomes in the short term and the long term. These outcomes may be visualized via solutions to the differential equations, or via the differential equations directly.

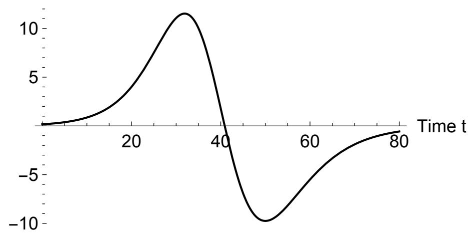
This activity focuses on interpreting graphical representations of both differential equations and their solutions.

Exercise 1: Initial Observations About the Differential Equations

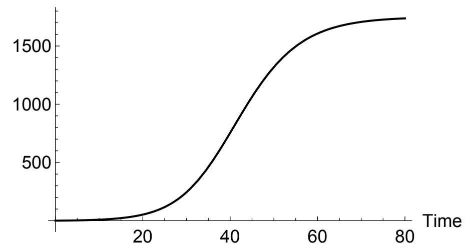
Consider the right-hand sides of the differential equations in the model above. Qualitatively, what can you say about $\frac{dS}{dt}$? Temporarily skip over $\frac{dI}{dt}$ and instead consider: what can you say about $\frac{dR}{dt}$? Finally, consider $\frac{dI}{dt}$. This differential equation includes both positive and negative terms. Can the right-hand side ever equal zero? What would this mean about the outbreak? Finally, be sure you think carefully about the interaction term βSI: when either S or I is close to zero, there cannot be many new infections per time unit, but when both S and I are larger, infection spreads quickly.

Exercise 2: Checking Understanding Via Graphs

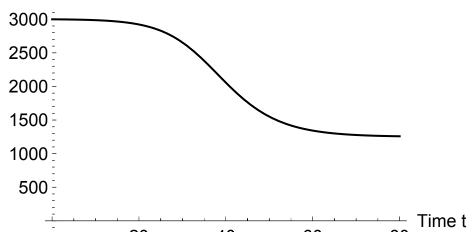
Below are six graphs from the SIR model for a theoretical outbreak of influenza in a population of 3,000 people. At the start of the outbreak, there is one Infectious person, and everyone else is Susceptible. There is one graph for each of the following: $S(t)$, $I(t)$, $R(t)$, dS/dt, dI/dt, and dR/dt. Which is which? How do you know? Consider shapes of graphs, values on the vertical axis, and other information you believe to be relevant. In particular, two graphs look nearly identical. One is a derivative graph (dS/dt, dI/dt, or dR/dt) and one is a solution graph ($S(t)$, $I(t)$, or $R(t)$). Reason through which graph is which. Explain your logic.



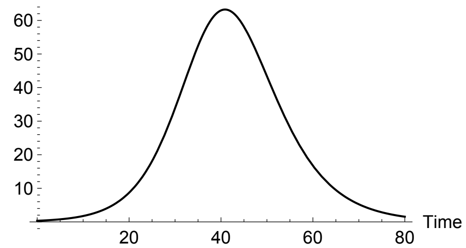
1a.)



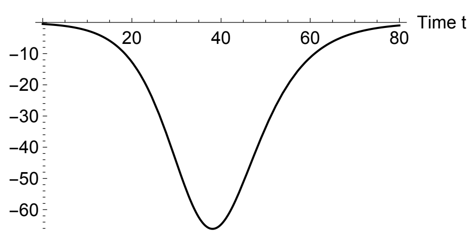
1b.)



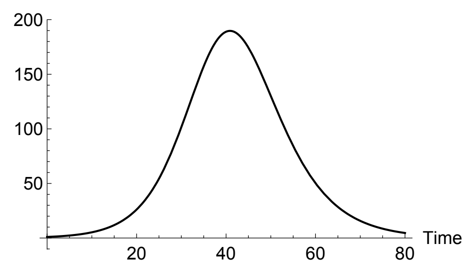
1c.)



1d.)



1e.)



1f.)

Exercise 3: Allow Discontinuities

Next imagine an outbreak in which β is not constant. Reasons for the change in β could include a public health response to the outbreak, where measures such as social distancing or hand washing may reduce the likelihood of disease spread. This can be modeled by reducing the value of β. Or, on a college campus, weekends or specific social events may cause a temporary increase in the value of β. Changes in β lead to changes in how many people become sick, and changes in the model provide a fascinating moment for building insight into the relationship between a differential equation and its solution.

As a first adjustment, consider $\beta = \beta(t)$ to be piecewise constant, as shown in Figure 2. This may represent suddenly changed health policies once authorities realize an outbreak has begun.

Again, determine which graph is which below. The choices are $S(t)$, $I(t)$, $R(t)$, dS/dt, dI/dt, and dR/dt.

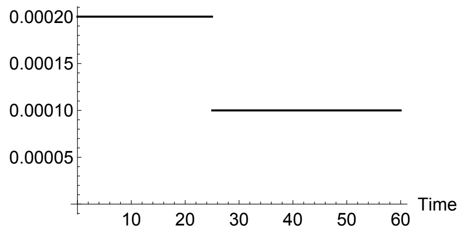
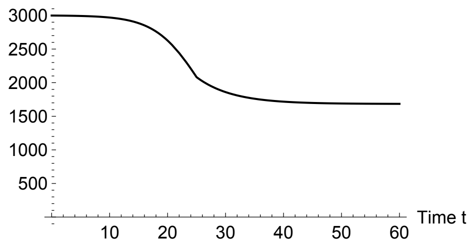
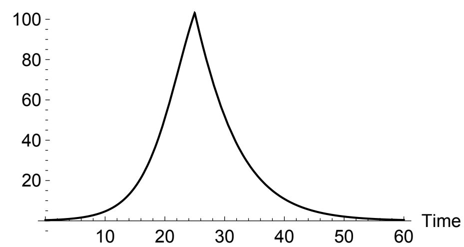
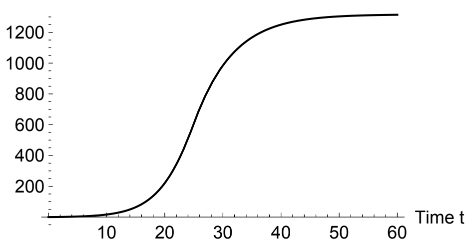
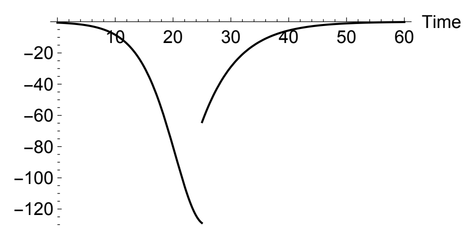


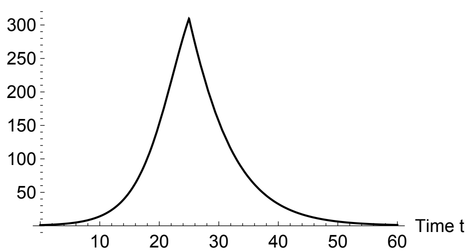
Figure 2. A piecewise beta function

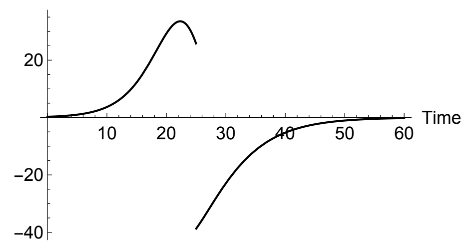
2a.)


2b.)


2c.)


2d.)


2e.)


2f.)


Exercise 4: Explore Further

Finally, try a more free-form approach.

- What would happen if $\beta(t)$ were constantly decreasing, that is, if $\beta(t)$ were a straight line with negative slope? (Be sure $\beta(t)$ remains nonnegative throughout the time of your outbreak.)
- What would happen if $\beta(t)$ were periodic? Can you construct a periodic function for $\beta(t)$, with a period of seven days, to represent weekly variation in infectivity? Again, keep $\beta(t) > 0$.
- What other $\beta(t)$ functions could you consider?

In all variations, think through what the graphs of $S(t)$, $I(t)$, $R(t)$, dS/dt, dI/dt, and dR/dt should look like. Reason through the answers first, and explain your reasoning carefully. Then, if you have software available for graphing differential equations and their solutions, test your claims by creating the graphs.