7-28-2020

Reviews

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Recommended Citation
Greer, M. L. "Reviews." (2020) Book Review. American Mathematical Monthly 127(7) 668-71. https://doi.org/10.1080/00029890.2020.1764825

500 Examples and Problems of Applied Differential Equations
Reviewed by Meredith L. Greer

The teaching and learning of differential equations present both vast opportunities and an intensive amount of decision making. How much emphasis should be on theory vs. on application? From which fields should examples be drawn? The landscape continues to change, and resources are crucial for both instructors and students. This review considers the new book "500 Examples and Problems of Applied Differential Equations" in these contexts and in relationship to other resources for studying differential equations.

Just a few decades ago, first courses in Differential Equations followed a fairly standard path, consisting primarily of analytic solutions for well-studied types of equations and systems. Course structures usually began with first-order differential equations before moving to second-order or higher-order equations, then to systems of differential equations, with separation of variables, integrating factors, power series, Laplace transforms, and other traditional closed-form solution approaches each emphasized in turn.

While this type of pathway through Differential Equations continues to exist, the past quarter century or so has seen a growing number of instructors shifting their focus. One reason is that computing power has changed what is possible. A second, and crucial, reason is that the real-world need for differential equations far exceeds the relatively few specific equations and systems that can be analytically solved via classical methods. As a result, many courses today emphasize applications, modeling, visualization, and graphical and numeric approaches to gleaning valuable information from differential equations [2]. Such curricular changes are strongly recommended by researchers studying undergraduate mathematics education [3, 5].

Textbooks for introductory courses in Differential Equations have adopted these curricular updates in a variety of ways. A few remain nearly unchanged from the texts of thirty years ago. Most retain a similar structure of mathematical topics while incorporating more applications. A few pursue modeling and visualization much more fully, pointedly jettisoning most analytic techniques to make room in the curriculum for computer-centered approaches to applied questions.

In all cases, instructors and students would do well to seek out additional applications of differential equations. This is a field with connections to, well, just about every aspect of our lives, not to mention a large array of student majors. A short list includes epidemiology, climate change, technology, economics, pharmacology, and engineering. We could go on and on, starting at mathematical history and not stopping before systems of differential equations representing romantic relationships. Indeed, the study of differential equations illustrates beautifully the power of mathematics to address seemingly disparate real-world phenomena using techniques that are, at their core, identical or similar.

This is the context in which "500 Examples and Problems of Applied Differential Equations" arrives. The collection of examples is indeed both large in number as well as impressively varied across disciplines.

On examination, it quickly becomes clear that this is not a standalone textbook, but instead serves as a supplement to other resources. Each chapter begins with a short introduction involving standard techniques or theorems. Several worked-through examples quickly follow, then a few dozen problems to be attempted. This suggests the question: who can most easily use this supplement?

The book's structure most closely maps onto a classical course in differential equations, making it ideal for instructors, and perhaps students, of these courses. The first several chapters focus on analytic solutions in standard differential equations categories such as first-order linear equations, first-order nonlinear equations, second- and higher order equations, and systems of first-order equations. Not till Chapter 6 does a numerical approach appear, fourth-order Runge-Kutta, the only numerical technique in this text. The following chapter describes equilibrium solutions and stability conditions. The remaining chapters discuss boundary-value problems, linear and nonlinear.

For instructors who teach Differential Equations with a focus on numerical and graphical solutions, and for students in such a course, "500 Examples…" requires adaptation. The book jumps in with integrating factors in Chapter 1, and integrating factors are one of the topics increasingly omitted from courses emphasizing more than analytic solutions [2]. Approximately the first half of the book discusses entirely analytic solutions. Graphical approaches appear mainly in the single chapter on stability conditions and are limited in scope.

Still, I found the wealth and variety of examples to be exciting. Here are just a few examples, spread across multiple chapters of the text.

Chapter 1, First-Order Linear Differential Equations: Radiocarbon dating relies on the common mathematical concept of exponential decay as it applies to the half-life of carbon-14. Its development resulted in a Nobel Prize. The mathematical approach makes a nice first-chapter example in Differential Equations, employing the equation $dN/dt = kN$, with $N(t)$ equal to the number of atoms of carbon-14 present at time t, and with k determined by the half-life. Given carbon-14's half-life of about 5730 years, radiocarbon dating does not work well for objects more than about 50,000 years old, but "500 Examples…" expands the options by providing a table of half-life times for over a dozen elements. Using this table, faculty and students can create examples of the more general idea of radiometric dating to identify the age of an even wider range of objects.

Chapter 2, Some First-Order Nonlinear Differential Equations: Examples in this chapter include the shape of a tsunami, with height measured as a function of distance from a specified offshore point; Stokes' law for the acceleration of spherical droplets due to gravity; and a representation of the brachistochrone problem, that is, finding the curve between two points along which an object descends most quickly.

Chapter 3, Second- and Higher-Order Differential Equations: Harmonic oscillators make up several exercises, starting with the classical combination of Hooke's law and Newton's second law. Another example is Samuelson's investment model in which $C(t)$ is capital and $I(t)$ investment, each at time t, and it is assumed that $dC/dt = I$. Further assumptions vary, with the basic model supposing $dI/dt = -m[C(t) - C_e]$ where C_e is an equilibrium level and m is a constant of proportionality. These equations result in a second-order differential equation.

Chapter 5, Systems of First-Order Differential Equations: Epidemics can be modeled using a system of equations, sometimes titled SIR, for Susceptible, Infectious, and Recovered or Removed populations. A 1927 paper by Kermack and McKendrick introduced this style of modeling, with people each belonging to a compartment (S, I, R, and possibly others) and one differential equation for each compartment. "500 Examples…" provides sample SIR systems, both linear and nonlinear. Analysis includes equilibrium solutions representing long-term outbreak outcomes, relationship of S with I, and mathematically-described circumstances in which the number of disease cases will increase or decrease.

Chapter 7, Stability Theory: An equilibrium solution of a differential equation system may be either stable—that is, it attracts solutions—or unstable, meaning it repels solutions. This chapter describes the details of determining stability, including eigenvalue formulations and diagrams of system behaviors. Illuminating examples ask the reader to determine stability in population dynamics, physics, and economics models (and more). The Lorenz equations also appear, permitting a foray into chaos theory while showing some of the difficulties in discussing stability.

Chapter 8, Linear Boundary Value Problems: One nice boundary value problem that gets substantial page space is that of the architectural columns in Greek and Roman structures. In particular: under what conditions might a column buckle? Equations developed by Euler are $y'' + \lambda y = 0$ and $y(0) = y(1) = 0$, where $y(x)$ is the column's lateral displacement and λ combines several physical properties of the column.

The assortment of examples is the primary reason to keep this text on hand as one source of ideas when teaching Differential Equations. The authors write, "Students would be more motivated to study the material of differential equations and retain it better if they are exposed to plenty of world applications," and I agree fully. The diversity of examples is the strongest reason to recommend this book.

Yet the emphasis on analytic solutions is restrictive. The authors also write, "Unfortunately, one of the major problems in differential equations is that it is usually impossible to obtain solutions to problems in closed (exact) form. As a result, one is forced to consider numerical methods…." An alternate, positive, framing is possible, embracing graphical and numerical solutions. Such solutions make differential equations applications accessible to students who are earlier in their mathematical training. Furthermore, the availability and continued development of graphical and numerical approaches contribute crucial differential equations knowledge to a slew of applied and complicated problems that cannot be well understood if we require or privilege analytic solutions.

Also worth noting, “500 Examples…” provides pre-established equations as models, but never asks readers to develop models. Given that there are approximately 500 examples in fewer than 400 pages, there is no space for this. The book uses minimal space for anything other than differential equations examples: introductions, explanations, and solution details are sparse.

Teaching students to develop, compare, and work with models has become an increasingly essential piece of education and curriculum in mathematics. Reports from within the mathematics profession, and communications from partner disciplines, urge mathematicians to include realistic modeling experiences for students. That is, instructors should help students learn how to build models that connect in some meaningful way to real-world circumstances and data, using both the well-studied tools of mathematics and the messiness of real situations. [2,3,4,5] Showing examples is helpful, but is not all that is needed: as noted in the 2016 GAIMME (Guidelines for Assessment and Instruction in Mathematical Modeling Education) report, “Applications are not necessarily modeling – there needs to be some autonomy and open-endedness.” [4]

Readers may wonder whether other resources exist for gathering and using differential equations examples while also incorporating modeling into courses. In other words, while “500 Examples…” provides ready-made applications of differential equations, do other resources provide modeling opportunities? The answer is: yes, and as a bonus, some of these are freely available! Two especially helpful resources are SIMIODE and the CODEE journal, described below.

SIMIODE (Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations) “provides traditional text material and many, many modeling scenarios to use as introduction and motivator for the rich study of differential equations through modeling and technology.” [6] Materials at the SIMIODE website are freely available and peer reviewed. Users can choose from incorporating individual modules into courses, to building an entire course from SIMIODE. Further involvement is possible via online community, annual modeling challenges, and submitting or reviewing modules and other materials [6].

The CODEE Journal (Community of Ordinary Differential Equations Educators) began in 1992 and offers open access peer reviewed articles and projects “that promote the teaching and learning of ordinary differential equations.” [1] Materials in the CODEE Journal may be Articles, which emphasize approaches to teaching and learning, or Projects, which are activities that can be directly used in differential equations classes [1]. Modeling is a key aspect of many materials in the CODEE Journal.

Both SIMIODE and the CODEE Journal are currently active and have a history of funding from the National Science Foundation. Each welcomes new submissions of material and new users seeking ideas for the teaching and learning of differential equations.

Given all these resource options, educators and students can choose what works best for them. A book on the shelf, such as "500 Examples…", is easy to reach for and browse. Online resources, such as SIMIODE and the CODEE Journal, are frequently updated and offer different search capabilities. In sum, this is a fabulous time to teach differential equations while incorporating context-based examples and modeling activities. We can motivate students and give our courses purpose by showing that mathematics plays a relevant and meaningful role in our daily lives and in the world around us.

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