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Exploring Structure in Origami Constructions

Deveena R. Banerjee

Exploring Structure in Origami Constructions

An Honors Thesis

Presented to the Faculty of the Department of Mathematics

Bates College

in partial fulfillment of the requirements for the

Degree of Bachelor of Science

by

Deveena Roy Banerjee

Lewiston, Maine

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Acknowledgments

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Introduction

In origami, an artist starts with some reference points, and creates folds along angles. Intersections of these folds form new reference points, and even more folds can be made. Looking at an unfolded origami crane, we can see how these reference points arise.

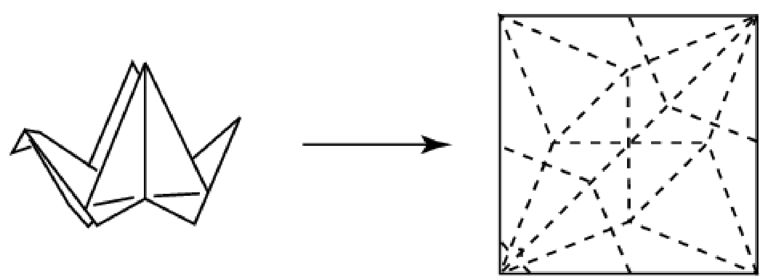


FIGURE 0.1. A folded and unfolded origami crane [**Hoi19**].

A question arises here about the structure of the intersection points. We turn this into a mathematical problem using the following criteria If we think of the paper as the complex plane, then we form a subset of this plane by intersecting along angles from seed points, and we are interested in special properties of this subset. The set of angles is prescribed for a given origami set. Under certain constraints, the origami construction gives rise to a subset with mathematical structure, including the topological structure of a lattice or the algebraic structure of a subring. In Chapter 1, we focus on some of the mathematical preliminaries that are necessary for understanding origami sets. This includes complex numbers and select topics from abstract algebra, and only those used in our discussion of origami sets. These select topics include the basics on groups, and culminates in group presentations and group actions. We do cover some of rings, but just to know when origami sets are rings. In Chapter 2, we explore the conditions that give rise to structured origami sets; these include origami lattices and origami rings. Much of this previous work on origami constructions focused on understanding the algebraic structure of origami sets. I added details to all proofs to ensure that I understood the leaps that were made in arriving at conclusions.

In Chapter 3, we turn our attention to the hyperbolic plane. The complex plane is an example of a Euclidean space, which is constructed using five axioms including the Parallel Postulate. When this axiom is removed from construction, we are left with the hyperbolic plane, leaving us with multiple parallel hyperbolic lines that can intersect a particular point. In our origami constructions, new reference points are made by intersecting two lines; two intersecting hyperbolic lines have the same properties as two intersecting Euclidean lines. Since the hyperbolic plane has fundamentally different geometry due to its axiomatization, the constructed points are different when we start with seed points in the hyperbolic plane compared to when they start in the complex plane.

Prior to this work, there was no defined hyperbolic origami construction. We make several attempts to go through the origami construction using ideas of hyperbolic geometry, and reach partial results.

We learn what a hyperbolic origami set is not. We use the same definitions of angles, but we have too many freedoms when it comes to the center and radius of a hyperbolic line. Our results of these do not produce reference points to use with our angles, suggesting that simply transferring to this new geometry with new rules is much more complicated than it appears. Then, we try a simple mapping from the upper half of an origami set in the complex plane to the Poincaré disk. However, the image of this map does not resemble tilings of the hyperbolic plane in the way that our complex origami lattices resemble tilings of the complex plane.

We then use the ideas of group action and the hyperbolic plane to classify origami lattices. We can classify homothetic complex lattices using a bijection to a point in the fundamental domain of the hyperbolic plane. Because all lattices, up to homothety, can be parameterized using the group action of the projective special linear group, we then explore what it means for origami lattices to be equivalent using many examples. We raise questions regarding the types of origami lattices that we can produce, and confer with previously determined results about classifications of origami sets. We also begin to explore whether all lattices can be produced using our origami constructions. We conjecture that all lattices are contained in origami lattices, and if a lattice Λ is not maximal, then the Λ is not an origami lattice.

CHAPTER 1

Some Mathematical Preliminaries

1. An Origami Set

Origami sets are constructed from a set of seed points and angles. Under certain constraints, the points constructed through this process are able to interact with each other using operations. These points are numbers, and under the aforementioned constraints, we can perform operations that produce different points that could also be constructed in the same process. We begin with an introduction to origami constructions, and then review ideas from complex analysis and abstract algebra that will allow us to understand the underlying algebraic structure of one of these constructions.

In origami, an artist folds along a piece of paper and uses the folds and resulting intersections to create a shape. Similarly, we concern ourselves with the intersections of lines formed along angles from certain seed points. Start with a set of points in $\mathbb{C}$, these are referred to as the "seed points" or "generator points". Extending from the points along the angles of a set U, we consider the point where two extensions intersect. The intersection point can be a new reference point that can be used to create a new intersection. All angles that we consider are on the unit circle. Note that both an angle α and $-\alpha$ are represented because any α represents the direction of a fold in both the forward and "backwards" directions, so we just include α in U. This phenomenon regarding the angles is shown in the following figure.

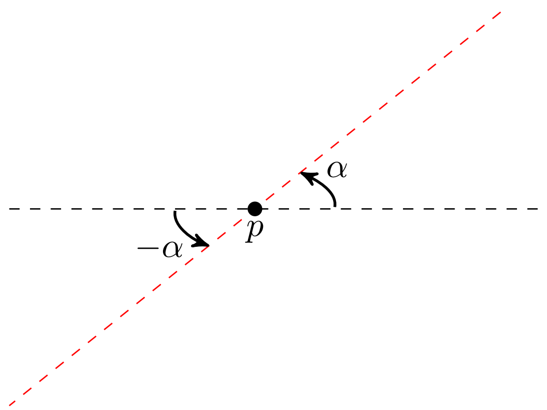


FIGURE 1.1. Extension along an angle α from a point p.

We can denote an intersection from a point p along angle $\alpha \in U$, and q along angle $\beta \in U$, by $[\![p, q]\!]_{\alpha,\beta}$. Each intersection point can now be used as a reference point, so an origami set is produced in generations. Each successive generation contains the previous generation, so we can express an origami set as the union of all generations:

$$M(U) = \bigcup_{k=0}^{\infty} M_k.$$

We borrow this notation from Möller [**Möl18**]. Note that expressing the origami set as a union of generations will allow us to use induction on the generations to show that each generation carries some property.

2. Complex Numbers

Since we start with seed points in $\mathbb{C}$, constructed points made from intersections are also complex numbers. Recall that complex numbers are combinations of real numbers that lie in the complex plane and take the form $z = a + bi$, that is, they have a real component and an imaginary unit, where $i = \sqrt{-1}$ [**BC04**]. As such, $i^2 = -1$. Graphically, a complex number is point that lies at the coordinate $(\text{Re } z, \text{Im } z)$, where $\text{Re } z = a$ and $\text{Im } z = b$. Then, we can think about the distance between z and 0, which is denoted and defined by $|z| = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2} = \sqrt{a^2 + b^2}$, and called the modulus of z. We can also think about the angle between the real axis and z, which we call the argument. The argument is given by $\arg(z) = \arctan\left(\frac{\text{Im } z}{\text{Re } z}\right)$. The principal argument, $\text{Arg}(z) = \Theta$ is the unique value such that $-\pi < \Theta \leq \pi$. We choose $\mathbb{C}$ to represent the plane encompassing our constructed points so that we may work with familiar numbers as constructed points, and may work with the pre-existing algebraic properties of $\mathbb{C}$.

We can add, subtract, multiply, and divide complex numbers in a very familiar way, where even commutativity exists for addition and multiplication. Details of these properties will be explored in the abstract algebra portion of this chapter. For two complex numbers, $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, we can perform our usual operations in a familiar way. First, let's review operations that we can perform with complex numbers.

Addition is defined in the following way:

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= (a_1 + a_2) + i\,(b_1 + b_2) \end{aligned}$$

The subtraction of complex numbers is given by:

$$\begin{aligned} z_1 - z_2 &= (a_1 + b_1 i) - (a_2 + b_2 i) \\ &= (a_1 - a_2) + i\,(b_1 - b_2)\,. \end{aligned}$$

Multiplication is defined using the distributive property:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + b_1 i) \cdot (a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + i b_1 a_2 + i^2 y_1 y_2 \\ &= (a_1 a_2 - b_1 b_2) + i\,(b_1 a_2 + a_1 b_2)\,. \end{aligned}$$

Division requires that we have real numbers in the denominator, and assuming that $z_2 \neq 0$:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + b_1 i}{a_2 + b_2 i} \\ &= \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} \\ &= \frac{a_1 a_2 - a_1 b_2 i + a_2 b_1 i + b_1 b_2}{(a_2)^2 + (b_2)^2} \\ &= \frac{a_1 a_2 + b_1 b_2}{(a_2)^2 + (b_2)^2} + \frac{a_2 b_1 - a_1 b_2}{(a_2)^2 + (b_2)^2} i. \end{aligned}$$

To ensure that there are only real numbers in the denominator, we use the complex conjugate of a complex number. Simply put, if $z = a + bi$, then the complex conjugate is $\overline{z} = a - bi$. Notice from our calculations in the quotient of two complex numbers that $z\overline{z} = a^2 + b^2 = |z|^2$. Conjugacy distributes over our operations, and below are properties that apply to all complex numbers. For any two complex numbers z_1 and z_2:

$$
\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, & \\
\overline{z_1 - z_2} &= \overline{z} - \overline{z_2}, & \\
\overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2, & \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}, & \text{if } z_2 \neq 0, \\
\overline{z} &= z, & \text{if and only if } z \in \mathbb{R}, \\
\overline{z^n} &= (\bar{z})^n, & n \in \mathbb{Z}, \\
|z|^2 &= z\bar{z} = \bar{z}z, & \\
\overline{\overline{z}} &= z. &
\end{aligned}
$$

Our angles in origami come from the unit circle. If we think of the unit circle on the complex plane, then the angles of the unit circle can be defined using their corresponding point on the unit circle. For example, the angle $\frac{\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. So, we say that $\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

The points made through origami construction are complex numbers. When we perform algebra or arithmetic on them, we are performing algebra and arithmetic on a subset of complex numbers. We will use these properties to understand the geometry that sets up our origami process, and use it to extract information about the sets of constructed points.

3. Groups and Rings

We will review concepts from Abstract Algebra so that we may understand the underlying algebra in origami sets. First, we will review some definitions and define what constitutes a "nice" structure. This following information can all be found in Gallian's text, *Contemporary Abstract Algebra* [**Gal09**].

We can *group* elements together into a set with an operations to study the interactions between elements. We will look at groups and rings, with aspects that will be important as we study the algebraic structure of origami sets.

Definition (Group). Let G be a set with a binary operation. We say that G is a *group* if it satisfies the following properties.

(1) The operation is associative in G, i.e. for all a, b, c in G, $a(bc) = (ab)c$.
(2) There is an identity element, e in G such that for all a in G, $ea = a = ae$.
(3) For every element a in G, there exists an element a^{-1} in G such that $aa^{-1} = e = a^{-1}a$.

Groups are the first level of structure that we will discuss. An example of a group that we will consider in the beginning of our origami ring exploration is $U = \{0, \frac{\pi}{3}, \frac{2\pi}{3}\}$. See that in U, addition is closed, and note that $\frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$, which can recognize as the extension along $\frac{\pi}{3}$ in the reverse direction of a particular point. Other examples of groups that are familiar sets include:

- the integers, $\mathbb{Z}$, under addition
- the positive rational numbers, $\mathbb{Q}^+$, under multiplication
- the integers mod n, $\mathbb{Z}_n$, under addition mod n,
- the general linear group of degree 2 over a field, $\mathrm{GL}_2(F)$, under matrix multiplication; this is the set of invertible 2×2 matrices with entries in a field F
- the dihedral group, D_n, under composition
- the special linear group of degree 2 over a field, $\mathrm{GL}_2(F)$, under matrix multiplication; this is the set of 2×2 matrices with entries in a field F and a determinant of 1
- the circle group, $\mathbb{T}$, the set of all complex numbers with modulus 1, under multiplication

Notice that some of the groups listed above carry an idea of commutativity with their operations. When the operation is commutative in a group, we call it an *abelian* group. Notice that the groups involving matrix multiplication are not abelian groups. Subsets of groups may form a group themselves, which we call a subgroup.

Definition (Subgroup). If a subset H of a group G is itself a group under the operation of G, we say that H is a subgroup of G, and denote the phenomenon as $H \leq G$.

How can we know if a particular subset is a subgroup? Aside from checking the parts of the definition of a group, we can also use a test to determine if a subset is a subgroup.

THEOREM 1.1 (One-Step Subgroup Test). *Let G be a group and H a nonempty subset of G. In multiplicative groups, if ab^{-1} is in H whenever a and b are in H, then H is a subgroup of G. In additive groups, if $a - b$ is in H whenever a and b are in H, then H is a subgroup of G.*

PROOF. We will show that these conditions show satisfaction of all parts of the definition of a group. The operation of H is the same as that of G, so the operation is associative. Since H is nonempty, choose the element $x \in H$. Let $a = x$, and $b = x$. So $e = xx^{-1} = ab^{-1} \in H$. Next, we must show that x^{-1} is in H whenever x is. Choose $a = e$ and $b = x$. We know that $ex^{-1} \in H$, so $x^{-1} \in H$ whenever x is. Finally, we'll show that H is closed under the operation of G. Let $a = x$ and $b = y^{-1}$. Then, $xy = ab^{-1}$, and H is closed under the operation.

$\square$

Next, we'll think about analyzing some groups using their cosets.

DEFINITION (Coset of H in G.). Let G be a group and let H be a subset of G. For any $a \in G$, the set $\{ah : h \in H\}$ is denoted by aH. Analogously, $Ha = \{ha : h \in H\}$ and $aHa^{-1} = \{aha^{-1} : h \in H\}$. When H is a subgroup of G, the set aH is called the left coset of H in G containing a, whereas Ha is called the right coset of H in G containing a.

The element a is called the coset representative of of aH or Ha. We use $|aH|$ to denote the number of elements in the set aH, and $|Ha|$ to denote the number of elements in Ha. Cosets are usually not subgroups. We'll look at an example of cosets of a known group: let

$H = \{0, 3, 6\}$ in $\mathbb{Z}_9$ under addition. The cosets of H in $\mathbb{Z}_9$ are:

$$0 + H = \{0, 3, 6\} = 3 + H = 6 + H$$

$$1 + H = \{1, 4, 7\} = 4 + H = 7 + H$$

$$2 + H = \{2, 5, 8\} = 5 + H = 8 + H.$$

Note that though $7 \neq 4 \neq 1$, they have the same left cosets. Further, left cosets and right cosets need not be equal to one another. There are some properties of cosets.

Proposition 1.2 (Properties of Cosets). *Let H be a subgroup of G, and let a and b belong to G. Then,*

(1) $a \in aH$,
(2) $aH = H$ *if and only if* $a \in H$,
(3) $aH = bH$ *if and only if* $a \in bH$,
(4) $aH = bH$ *or* $aH \cap bH = \emptyset$,
(5) $aH = bH$ *if and only if* $a^{-1}b \in H$,
(6) $|aH| = |bH|$,
(7) $aH = Ha$ *if and only if* $H = aHa^{-1}$,
(8) aH *is a subgroup of* G *if and only if* $a \in H$.

Proof. We'll prove each item individually.

(1) Consider $a = ae$, which we know to be an element of aH.
(2) Suppose that $aH = H$. Then $a = ae \in aH = H$. Now, assume $a \in H$, and we'll show that $aH \subseteq H$ and $H \subseteq aH$. Closure of the operation implies $aH \subseteq H$. We"ll show that $H \subseteq aH$; consider $h \in H$. Since $a \in H$, and $h \in H$, we know $a^{-1}h \in H$. So, $h = eh = (aa^{-1})\, h = a\,(a^{-1}h) \in aH$.

(3) If $aH = bH$, then $a = ae \in aH = bH$. Conversely, if $a \in bH$ we have $a = bh$ where $h \in H$, and therefore $aH = (bh)H = b(hH) = bH$.

(4) If there is an element $c \in aH \cap bH$, then $cH = aH$ and $cH = bH$.

(5) From Proposition 1.2.2, we know $aH = H$ if and only if $a \in H$. Additionally, $aH = bH$ if and only if $H = a^{-1}bH$. Combining these two facts shows that $aH = bH$ if and only if $a^{-1}b \in H$.

(6) We can define a one-to-one map from aH onto bH. This map is one-to-one from the cancellation law.

(7) See that $aH = Ha$ if and only if $(aH)a^{-1} = (Ha)a^{-1} = H\left(aa^{-1}\right) = H$, which is true only if $aHa^{-1} = H$.

(8) If aH is a subgroup of G, then aH contains e. So, $aH \cap eH \neq \emptyset$. Then, $aH = eH = H$, so $a \in H$. Conversely, if $a \in H$, then $aH = H$.

$\square$

Let's return to the question of when left and right cosets are equal to one another. This is only true in some situations, which have immense importance.

Definition (Normal Subgroup). A subgroup H of a group G is called a *normal subgroup* of G if $aH = Ha$ for all a in G. We denote this by $H \lhd G$.

Note that this does *not* mean that $ah = ha$ for $a \in G$ and $h \in H$. To determine if a particular subgroup is a normal subgroup, we can use the following theorem.

Theorem 1.3 (Normal Subgroup Test). *A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.*

Proof. If H is normal in G, then for any $x \in G$, and $h \in H$, there is an h' in H such that $xh = h'x$. Thus, $xhx^{-1} = h'$ and as such, $xHx^{-1} \subseteq H$. Conversely, if $xHx^{-1} \subseteq H$ for all x, then let $x = a$. We then have $a^{-1}H\left(a^{-1}\right)^{-1} = a^{-1}Ha \subseteq H$. So, $Ha \subseteq aH$.

□

When the subgroup H of G is normal, then the left or right cosets of H in G is itself a group. We call this the factor, or quotient, group of G by H.

Definition (Factor Group). Let G be a group, and let H be a normal subgroup of G. The set $G/H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = abH$.

We can think of factor groups as "mod"-ing out by a particular coset. Now, let's move onto the sets with more than one binary operation. Sometimes, the set of constructed points of our origami set are a ring, which allows us to compare different origami sets.

Definition (Ring). Let R be a set with two binary operations, addition and multiplication. We say that R is a *ring* if for all a, b, and c in R, the following are satisfied:

(1) $a + b = b + a$.
(2) $(a + b) + c = a + (b + c)$.
(3) There is an additive identity, 0, such that $a + 0 = a$ for all a in R.
(4) There is an element $-a$ in R such that $a + (-a) = 0$.
(5) $(ab)c = a(bc)$.
(6) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

When the multiplication is commutative, we say that R is a *commutative* ring. Some rings may have an element that is the identity under multiplication, which we call the *unity* in ring. For rings with a unity, elements that have a multiplicative inverse are called *units* in R.

Some examples of familiar sets that are rings include:

- the set of integers, $\mathbb{Z}$,
- the set of rational numbers, $\mathbb{Q}$,
- the set of real numbers, $\mathbb{R}$,
- the set of complex numbers, $\mathbb{C}$,
- the set of 2×2 matrices with integer entries.

Note that in each of these set, there are familiar notions of addition and multiplication. These sets are also all groups under addition. In fact, the defined addition and multiplication of a ring in these sets follows our usual, pre-conceived notions of addition and multiplication.

Definition (Subring). A subset S of a ring R is a *subring* if S itself is a ring under the operations of R.

Thinking about our familiar rings, we can identify some examples, as well as a non-example, of subrings.

- For the integers, $\mathbb{Z}$ as a ring R,
  - the even integers, $2\mathbb{Z}$ form a subring S of $\mathbb{Z}$.
  - the odd integers do not form a ring; while an odd integer multiplied by an odd integer is certainly odd, the addition of two odd integers yields an even integer. Hence, the odd integers are not a subgroup of $\mathbb{Z}$ under addition.
- For the complex numbers, $\mathbb{C}$ as R,
  - the set of Gaussian integers, $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ form a subring S of $\mathbb{C}$.
- For the real numbers, $\mathbb{R}$ as R,
  - the rational numbers, $\mathbb{Q}$, form a subring S of $\mathbb{R}$.
- For the ring of 2×2 matrices with integer entries, $M_2(\mathbb{Z})$,
  - the set of 2×2 diagonal matrices with integer entries, $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{Z} \right\}$, is a subring S of $M_2(\mathbb{Z})$,

There is a simple test to see if a subset of a ring is a subring.

Theorem 1.4 (Subring Test). *A nonempty subset S of a ring R is a subring if S is closed under subtraction and multiplication.*

Proof. Since addition in R is commutative and S is closed under subtraction, by the One-Step Subgroup Test, S is an abelian group under addition. Since multiplication in R is associative and distributive over addition, the same must be true for multiplication in S. So, the only condition that must be checked is that multiplication is a binary operation on S, which shows closure under multiplication.

$\square$

Polynomials are fairly familiar objects, and we'll add them to our repertoire of algebraic objects. Much of the work done in developing the original ideas for this thesis do not deal with polynomial rings, but they are essential in the previous work done on origami rings.

DEFINITION (Ring of Polynomials over R). Let R be a commutative ring. The set

$$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_i \in R, n \text{is a non-negative integer}\}$$

is called the *ring of polynomials over R in the indeterminate x*. Two elements

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

of $R[x]$ are considered equal if and only if $a_i = b_i$ for all non-negative integers i.

Addition and multiplication of polynomials in $R[x]$ involve collecting like terms. The factoring of polynomial rings allows for the classification of polynomials and hence rings. We utilize them to prove one condition of an origami ring.

4. Group Presentations and Group Action

There are ways to define a group with certain prescribed properties. We begin with a set of elements that generate the group and a set of equations that specify the conditions that these generators must satisfy. Such a presentation will uniquely determine a group up to isomorphism.

Definition (Words from S). For a set $S = \{a, b, c, \dots\}$, of distinct symbols, we create a new set $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}, \dots\}$ by replacing each x in S by x^{-1}. The set $W(S)$ is the collection of all formal finite strings of the form $x_1 x_2 \cdots x_k$, where eeach $x_i \in S \cup S^{-1}$. The elements of $W(S)$ are called *words from S*.

The empty word, denoted by e, is the string with no elements, and the empty word is in $W(S)$.

We can define a binary operation on the set $W(S)$ by juxtaposition. If $x_1 x_2 \cdots x_k$ and $y_1 y_2 \cdots y_t$ belong to $W(S)$, then $x_1 x_2 \cdots x_k y_1 y_2 \cdots y_t$ is in $W(S)$. This operation is associative, and the empty word is the identity. Note that a word aa^{-1} is not the identity, because the elements of $W(S)$ are formal symbols with no implied meaning. This presents a problem, because we cannot make a group out of $W(S)$. Instead, we may define equivalence classes of words.

To determine whether two things are the same under a certain context, recall that we use equivalence relations.

Definition (Equivalence Relation). An *equivalence relation* on a set S is a set R of pairs of elements of S such that:

(1) $(a, a) \in R$ for all $a \in S$.
(2) $(a, b) \in R$ implies $(b, a) \in R$.
(3) $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

An equivalence relation partitions a set into equivalence classes. Two elements of a set are equivalent if they belong to the same equivalence class. Words can be classified by equivalence classes.

Definition (Equivalence Classes of Words). For any pair of elements u and v of $W(S)$, we say that u is related to v if v can be obtained from u by a finite sequence of insertions or deletions of words of the form xx^{-1} or $x^{-1}x$, where $x \in S$.

This relation is an equivalence relation on $W(S)$. We know that u is related to u because u is obtained from itself by not performing insertions or deletions. If v can be obtained from u by inserting or deleting words of the form xx^{-1} or $x^{-1}x$, then u can be obtained from v by deleting or inserting words of the form xx^{-1} or $x^{-1}x$ (reversing the procedure for obtaining v from u). If u can be obtained from v, and v can be obtained from w, then u can be obtained from w by obtaining first v from w, and then u from v. Hence we have shown the three parts of an equivalence relation for this relation on $W(S)$.

For an example of this, let's consider the set $S = \{a, b, c\}$. Then the word $acc^{-1}b$ is equivalent to ab because we delete the word cc^{-1}. Additionally, the word $aab^{-1}bbaccc^{-1}$ is equivalent to $aabac$; the word

$a^{-1}aabb^{-1}a^{-1}$ is equivalent to the empty word; $ca^{-1}b$ is equivalent to $cc^{-1}caa^{-1}bbca^{-1}ac^{-1}b^{-1}$. However, $cac^{-1}b$ is *not* equivalent to ab.

Definition (Free Group on S). Let S be a set of distinct symbols. For any word u in $W(S)$, let $\overline{u}$ denote the set of all words in $W(S)$ equivalent to u. Then the set of all equivalence classes of elements of $W(S)$ is the free group on S, under the operation $\overline{u} \cdot \overline{v} = \overline{uv}$. The empty word is the identity. If $w = x_1 x_2 \cdots x_k$, then $w^{-1} = x_k^{-1} x_{k-1}^{-1}, \cdots, x_2^{-1}, x_1^{-1}$.

We'll use the definition of a free group to show that every group can be related back to a free group.

Theorem 1.5 (Universal Mapping Property). *Every group is a homomorphic image of a free group.*

Proof. Let G be a group, and let S be a set of generators for G. Note that S may be G itself. Let F be the free group on S. Denote the word $x_1 x_2 \cdots x_n$ in $W(S)$ by $(x_1 x_2 \cdots x_n)_F$, and the product $x_1 x_2 \cdots x_n$ in G with $(x_1 x_2 \cdots x_n)_G$. Recall that $\overline{x_1 x_2 \cdots x_n}$ is different than $(x_1 x_2 \cdots x_n)_G$, because the operations of F and G are different. Consider the map from F into G given by

$$\phi\left(\overline{x_1 x_2 \cdots x_n}\right) = (x_1 x_2 \cdots x_n)_G .$$

See that ϕ is well-defined. Inserting or deleting expressions of the form xx^{-1} or $x^{-1}x$ in elements of $W(S)$ corresponds to inserting or deleting the identity in G. Also see that ϕ is operation-preserving:

$$\phi\left(\left(\overline{x_1x_2\cdots x_n}\right)\left(\overline{y_1y_2\cdots y_m}\right)\right) = \phi\left(\overline{x_1x_2\cdots x_ny_1y_2\cdots y_m}\right)$$

$$\phi\left(\left(\overline{x_1x_2\cdots x_n}\right)\left(\overline{y_1y_2\cdots y_m}\right)\right) = \left(x_1x_2\cdots x_ny_1y_2\cdots y_m\right)_G$$

$$\phi\left(\left(\overline{x_1x_2\cdots x_n}\right)\left(\overline{y_1y_2\cdots y_m}\right)\right) = \left(x_1x_2\cdots x_n\right)_G\left(y_1y_2\cdots y_m\right)_G$$

Finally, ϕ is onto G because S generates G. □

We have the background for defining a group by generators and relations.

Definition (Generators and Relations). Let G be a group generated by some subset $A = \{a_1, a_2, \ldots, a_n\}$ and let F be the free group on A. Let $W = \{w_1, w_2, \ldots, w_t\}$ be a subset of F and let N be the smallest normal subgroup of G containing W. We say that G is given by the generators $a_1, a_2, \ldots, a_n$ and the relations $w_1 = w_2 = \cdots = w_t = e$ if there is an isomorphism from F/N onto G that carries a_iN to a_i.

We denote a group's generators and relations by:

$$G = \langle a_1, a_2, \ldots, a_n | w_1 = w_2 = \cdots = w_t = e\rangle.$$

Then, we say that G has the *group presentation*

$$\langle a_1, a_2, \ldots, a_n | w_1 = w_2 = \cdots = w_t = e\rangle.$$

To count the number of objects that are considered nonequivalent, we can use a homomorphism.

DEFINITION (Group Action). If G is a group and S is a set of objects, then G acts on S if there is a homomorphism γ from G to $\mathrm{sym}(S)$, the group of all permutations on S. This homomorphism is the group action of G on S.

We denote the image of G under γ as γ_g. Two objects x and y in S are viewed as equivalent under the action of G if and only if $\gamma_g(x) = y$ for some g in G. When γ is one-to-one, the elements of G are permutations on S. If γ is not one-to-one, then there are distinct elements $g, h \in G$ such that γ_g and γ_h induce the same permutation on S.

With this background on complex numbers and abstract algebra, we are ready to explore origami sets.

CHAPTER 2

Introduction to Origami Constructions

1. The Intersection Operator

To understand the algebraic connections to origami constructed sets, we must first understand some properties of the intersections. Since each construction is an intersection problem, rather than going through the step-by-step construction, we can calculate each particular point. Recall that our goal is to generalize origami. For α and β, distinct directions, let L_α and L_β denote the lines along these directions, respectively. Here, we'll think about our angles as complex numbers using their point on the unit circle. Then $[\![p,q]\!]_{\alpha,\beta}$ represents the point z where $z = p + r\alpha = q + s\beta$, where there is a unique solution (r, s), $r, s \in \mathbb{R}$. Solving for s yields $s = \beta^{-1}(p - q + r\alpha)$. Since $r, s \in \mathbb{R}$, $\operatorname{Im} s = 0$ implies that from $s = \left(\frac{p-q}{\beta} + r\frac{\alpha}{\beta}\right)$, $r = \frac{\operatorname{Im}\left(\frac{p-q}{\beta}\right)}{\operatorname{Im}\left(\frac{\alpha}{\beta}\right)}$. So,

$$[\![p,q]\!]_{\alpha,\beta} = p + \frac{\operatorname{Im}\left(\frac{p-q}{\beta}\right)}{\operatorname{Im}\left(\frac{\alpha}{\beta}\right)}\alpha \tag{2.1}$$

$$= \frac{\alpha\overline{p}\beta - \overline{\alpha}p\beta - \beta\overline{q}\alpha + \overline{\beta}q\alpha}{\alpha\overline{\beta} - \overline{\alpha}\beta} \tag{2.2}$$

$$= \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{-\beta\overline{q} + \overline{\beta}q}{\alpha\overline{\beta} - \overline{\alpha}\beta}\alpha \tag{2.3}$$

$$= \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha. \tag{2.4}$$

Between Equation 2.3 and Equation 2.4, we use facts about the conjugates of complex numbers. Note that by using the complex conjugate of an angle, we must use the representation of angles as complex numbers.

Let's see how this works with an example of an origami set. Consider the the points $\{0, 1\} \in \mathbb{C}$ and the angles from $\left\{0, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$. Starting with the seed points, we extend along these prescribed angles, and note intersection points. These intersection points can then be used as new reference points. In this example, we produce $\mathbb{Z}\left[\zeta_3\right]$, the integer combinations of the third roots of unity. We will go through this stepwise construction. Note that we may also think of angles as points on the unit circle lying in $\mathbb{C}$, and hence the angles may also be represented by $\left\{1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\}$. We will return with a deeper exploration of lattices and the choice in angles later.

FIGURE 2.1. The seed points.

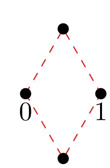


FIGURE 2.2. We begin to construct some points: $[\![0, 1]\!]_{\frac{\pi}{3}, \frac{2\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $[\![0, 1]\!]_{\frac{2\pi}{3}, \frac{\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

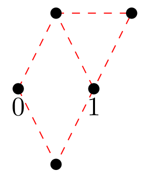


FIGURE 2.3. We continue with this process: $\left[\!\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i, 1\right]\!\right]_{0,\frac{\pi}{3}} = \frac{3}{2}+\frac{\sqrt{3}}{2}i.$

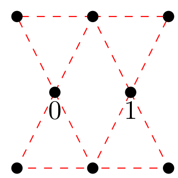


FIGURE 2.4. The next few points are: $\left[\!\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i, 0\right]\!\right]_{0,\frac{\pi}{3}} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$, $\left[\!\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i, 1\right]\!\right]_{0,\frac{2\pi}{3}} = \frac{3}{2}-\frac{\sqrt{3}}{2}i$, and $\left[\!\left[0, \frac{1}{2}+\frac{\sqrt{3}}{2}i\right]\!\right]_{\frac{2\pi}{3},0} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i.$

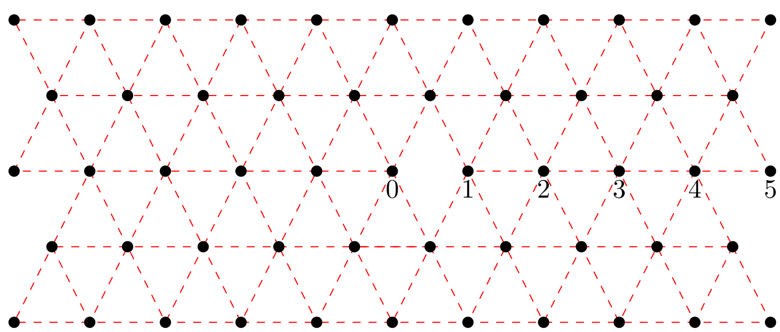


FIGURE 2.5. After an infinite number of generations, this is the construction, which may remind a reader of the chemical structure of graphene.

From the general form of an intersection point, we can state some facts about a point constructed from p and q from angles α and β that will allow us to move towards understanding the underlying structure and further generalizations of origami constructions. Much of this work comes from Buhler, Butler, De Launey, and Graham [**BBDLG12**] and Möller [**Möl18**]. While much of this is not new work, I rewrote the proofs in my words and have added many of the steps and details of them.

Proposition 2.1 (Properties of Intersection Points). *For p and q, points in the plane, and α and β, distinct angles in U, the following statements hold.*

(1) *A constructed point is symmetric:* $[\![p,q]\!]_{\alpha,\beta} = [\![q,p]\!]_{\beta,\alpha}$.

(2) *A constructed point can be reduced to a sum:* $[\![p,q]\!]_{\alpha,\beta} = [\![p,0]\!]_{\alpha,\beta} + [\![0,q]\!]_{\alpha,\beta}$.

(3) *The constructed point $[\![p,0]\!]_{\alpha,\beta}$ is a projection of p onto the line $r\beta, r \in \mathbb{R}$, in the direction of α.*

(4) *Constructed points are linear in one coordinate:* $[\![p+q,0]\!]_{\alpha,\beta} = [\![p,0]\!]_{\alpha,\beta} + [\![q,0]\!]_{\alpha,\beta}$*, and for any* $r \in \mathbb{R}$, $[\![rp,0]\!]_{\alpha,\beta} = r[\![p,0]\!]_{\alpha,\beta}$.

(5) *A constructed point $[\![p,q]\!]_{\alpha,\beta}$ has the form $Ap + Bq$, where A and B are real-linear maps of the complex plane that satisfy $A + B = 1_{\mathbb{C}}$.*

(6) *A constructed point can be rotated for an $\omega \in \mathbb{T}$, where $\mathbb{T}$ is the circle group:* $\omega[\![p,q]\!]_{\alpha,\beta} = [\![\omega p,\omega q]\!]_{\omega\alpha,\omega\beta}$.

PROOF. These facts can be proved using the expression for the intersection point $[\![p, q]\!]_{\alpha,\beta}$ in Equation 2.4 above and the principles of the construction.

(1) Using Equation 2.4, we know that

$$[\![p, q]\!]_{\alpha,\beta} = \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha.$$

Then,

$$\begin{aligned}
[\![q, p]\!]_{\beta,\alpha} &= \frac{\beta\overline{q} - \overline{\beta}q}{\beta\overline{\alpha} - \overline{\beta}\alpha}\alpha + \frac{\alpha\overline{p} - \overline{\alpha}p}{\overline{\beta}\alpha - \beta\overline{\alpha}}\beta \\
&= \frac{\beta\overline{q} - \overline{\beta}q}{\beta\overline{\alpha} - \overline{\beta}\alpha}\alpha + \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta \\
&= \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha \\
&= [\![p, q]\!]_{\alpha,\beta}.
\end{aligned}$$

(2) Using Equation 2.4,

$$[\![p, 0]\!]_{\alpha,\beta} = \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta(0) - \overline{\beta}(0)}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha = \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta,$$

and

$$[\![0, q]\!]_{\alpha,\beta} = \frac{\alpha(0) - \overline{\alpha}(0)}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha = \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha.$$

So,

$$\begin{aligned}
[\![p, 0]\!]_{\alpha,\beta} + [\![0, q]\!]_{\alpha,\beta} &= \frac{\alpha\overline{p} - \overline{\alpha}p}{\alpha\overline{\beta} - \overline{\alpha}\beta}\beta + \frac{\beta\overline{q} - \overline{\beta}q}{\overline{\alpha}\beta - \alpha\overline{\beta}}\alpha \\
&= [\![p, q]\!]_{\alpha,\beta}.
\end{aligned}$$

(3) The fact that $[\![p, 0]\!]_{\alpha,\beta}$ is a projection of p onto a line in the direction α is a consequence of the construction. For a point $[\![p, 0]\!]_{\alpha,\beta}$, let $\delta\left([\![p, 0]\!]_{\alpha,\beta}\right)$ represent the intersection of the real

axis with the line of slope δ through $[\![p,0]\!]_{\alpha,\beta}$.

(4) By Equation 2.4, $[\![p+q,0]\!]_{\alpha,\beta} = \dfrac{\alpha\left(\overline{p+q}\right)-\overline{\alpha}\left(p+q\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta$. Then,

$$\begin{aligned}
[\![p,0]\!]_{\alpha,\beta}+[\![q,0]\!]_{\alpha,\beta} &= \frac{\alpha\overline{p}-\overline{\alpha}p}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta+\frac{\alpha\overline{q}-\overline{\alpha}q}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= \frac{\alpha\overline{p}-\overline{\alpha}p+\alpha\overline{q}-\overline{\alpha}q}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= \frac{\alpha\left(\overline{p}+\overline{q}\right)-\overline{\alpha}\left(p+q\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= \frac{\alpha\left(\overline{p+q}\right)-\overline{\alpha}\left(p+q\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= [\![p+q,0]\!]_{\alpha,\beta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\![rp,0]\!]_{\alpha,\beta} &= \frac{\alpha\left(\overline{rp}\right)-\overline{\alpha}\left(rp\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= \frac{\alpha\left(\overline{r}\right)\left(\overline{p}\right)-\overline{\alpha}\left(r\right)\left(p\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= \frac{\alpha r\overline{p}-\overline{\alpha}rp}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= r\frac{\alpha\overline{p}-\overline{\alpha}\left(p\right)}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta \\
&= r[\![p,0]\!]_{\alpha,\beta}.
\end{aligned}$$

(5) By Proposition 2.1.2, $[\![p,q]\!]_{\alpha,\beta} = [\![p,0]\!]_{\alpha,\beta}+[\![0,q]\!]_{\alpha,\beta}$. Then, by 2.1.3, we know that $[\![p,0]\!]_{\alpha,\beta} = (p+r\alpha)$, a linear map of p. Similarly, $[\![0,q]\!]_{\alpha,\beta} = (q+s\beta)$, a linear map of q. So, $[\![p,q]\!]_{\alpha,\beta} = Ap+Bq$.

(6) By Equation 2.4,

$$\omega[\![p,q]\!]_{\alpha,\beta} = \omega\left(\frac{\alpha\overline{p}-\overline{\alpha}p}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta + \frac{\beta\overline{q}-\overline{\beta}q}{\overline{\alpha}\beta-\alpha\overline{\beta}}\alpha\right).$$

Now,

$$\begin{aligned}
[\![\omega p,\omega q]\!]_{\omega\alpha,\omega\beta} &= \frac{(\omega\alpha)\left(\overline{\omega p}\right)-\left(\overline{\omega\alpha}\right)(\omega p)}{(\omega\alpha)\left(\overline{\omega\beta}\right)-\left(\overline{\omega\alpha}\right)(\omega\beta)}(\omega\beta) + \frac{(\omega\beta)\left(\overline{\omega q}\right)-\left(\overline{\omega\beta}\right)(\omega q)}{\left(\overline{\omega\alpha}\right)(\omega\beta)-(\omega\alpha)\left(\overline{\omega\beta}\right)}(\omega\alpha) \\
&= \frac{\omega\alpha\bar{\omega}\bar{p}-\bar{\omega}\bar{\alpha}\omega p}{\omega\alpha\bar{\omega}\bar{\beta}-\bar{\omega}\bar{\alpha}\omega\beta}\omega\beta + \frac{\omega\beta\bar{\omega}\bar{q}-\bar{\omega}\bar{\beta}\omega q}{\bar{\omega}\bar{\alpha}\omega\beta-\omega\alpha\bar{\omega}\bar{\beta}}\omega\alpha \\
&= \frac{\left(\omega\overline{\omega}\right)\alpha\overline{p}-\left(\overline{\omega}\omega\right)\overline{\alpha}p}{\left(\omega\overline{\omega}\right)\alpha\overline{\beta}-\left(\overline{\omega}\omega\right)\overline{\alpha}\beta}\omega\beta + \frac{\left(\omega\overline{\omega}\right)\beta\overline{q}-\left(\overline{\omega}\omega\right)\overline{\beta}q}{\left(\overline{\omega}\omega\right)\overline{\alpha}\beta-\left(\omega\overline{\omega}\right)\alpha\overline{\beta}}\omega\alpha \\
&= \frac{\omega\overline{\omega}\alpha\overline{p}-\omega\bar{\omega}\bar{\alpha}p}{\omega\overline{\omega}\alpha\overline{\beta}-\omega\bar{\omega}\bar{\alpha}\beta}\omega\beta + \frac{\omega\overline{\omega}\beta\overline{q}-\omega\bar{\omega}\bar{\beta}q}{\omega\bar{\omega}\bar{\alpha}\beta-\omega\overline{\omega}\alpha\overline{\beta}}\omega\alpha \\
&= \frac{\alpha\overline{p}-\overline{\alpha}p}{\alpha\overline{\beta}-\overline{\alpha}\beta}\omega\beta + \frac{\beta\overline{q}-\overline{\beta}q}{\overline{\alpha}\beta-\alpha\overline{\beta}}\omega\alpha \\
&= \omega\left(\frac{\alpha\overline{p}-\overline{\alpha}p}{\alpha\overline{\beta}-\overline{\alpha}\beta}\beta + \frac{\beta\overline{q}-\overline{\beta}q}{\overline{\alpha}\beta-\alpha\overline{\beta}}\alpha\right) \\
&= \omega[\![p,q]\!]_{\alpha,\beta}.
\end{aligned}$$

□

2. Expressing the Origami Set

Similar to how we can express complex numbers z as a linear combination of a real number and an imaginary number, $z = a + bi$, we should try to find a simple way to express our origami construction. Additionally, note that points of an origami construction fulfill the following equation:

$$[\![r,s]\!]_{\alpha,\beta} = \left(r + \mathbb{R}e^{i\alpha}\right) \cap \left(s + \mathbb{R}e^{i\beta}\right).$$

DEFINITION. A *monomial* is a point p in $M(U)$ that can be constructed in one step from a generator.

Any monomial p lies in a sequence where

$$\begin{aligned} p_1 &= [\![1,0]\!]_{\alpha_1,\beta_1} \\ p_2 &= [\![p_1,0]\!]_{\alpha_2,\beta_2} \\ &\vdots \\ p_{k-1} &= [\![p_{k-2},0]\!]_{\alpha_{k-1},\beta_{k-1}} \\ p_k &= [\![p_{k-1},0]\!]_{\alpha_k,\beta_k} = p. \end{aligned}$$

The integer k is the *length* of a monomial.

THEOREM 2.2. *The product of two monomials is a monomial.*

PROOF. Given $\alpha_1, \alpha_2, \beta_1,$ and β_2 in U, let $r = \frac{[\alpha_1,1]}{[\alpha_1,\beta_2]} \in \mathbb{R}$. Then $[\![0,1]\!]_{\alpha_1,\beta_1} = rv$, and by the rotation and linearity properties of points in M,

$$\begin{aligned} [\![0,1]\!]_{\alpha_1,\beta_1}[\![0,1]\!]_{\alpha_2,\beta_2} &= rv[\![0,1]\!]_{\alpha_2,\beta_2} \\ &= r[\![v,0]\!]_{(\beta_1\alpha_2),(\beta_1\beta_2)} \\ &= [\![rv,0]\!]_{(\beta_1\alpha_2),(\beta_1\beta_2)} \\ &= [\![[\![0,1]\!]_{\alpha_1,\beta_1},0]\!]_{(\beta_1\alpha_2),(\beta_1\beta_2)}. \end{aligned}$$

Hence we have proven that the product of monomials is a monomial.

□

With Proposition 2.1, we can begin to think about possible conditions that give rise to algebraic structure in the construction, namely, when the origami set is a ring. We start with the first constraint on an origami ring, which has to do with the angles.

THEOREM 2.3. *If U is a group of directions that determine at least three folds, then $M(U)$ is the set of integral linear combinations of $U-$monomials, and is therefore a subring of the complex numbers.*

PROOF. Let's define S to be the set of integral linear combinations of $U-$monomials. Note that S is an additive group due to the construction. We know that the product of two monomials is a monomial, as proved in Theorem 2.2, so S is a ring. Now, clearly $M(U)$ contains all possible monomials, and due to the construction, is also an additive group. So, S is contained in $M(U)$. By the linearity of points in M, as proven in Proposition 2.1, an element $r \in M$ has the form $r = [\![p, q]\!]_{\alpha,\beta} = [\![p, 0]\!]_{\alpha,\beta} + [\![0, q]\!]_{\alpha,\beta}$. If p and q can be expressed as integral linear combinations of monomials of length less than or equal to k, then r is an integral linear combination of monomials of at most $k+1$:

$$r = [\![p, q]\!]_{\alpha,\beta} = [\![p, 0]\!]_{\alpha,\beta} + [\![0, q]\!]_{\alpha,\beta} = [\![p_{k-1}, 0]\!]_{\alpha_{k-1},\beta_{k-1}} + [\![0, q_{k-1}]\!]_{\alpha_{k-1},\beta_{k-1}}.$$

This shows that S is contained in $M(U)$, and $M(U)$ is a subring of the complex numbers, completing our proof.

□

With all of this information, we can continue to explore origami constructions. A second example uses the same generator points, $\{0, 1\} \in \mathbb{C}$, but with the angles $U = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$. This stepwise construction is shown below, and we will continue to refer to this example as well as the previous triangular tiling throughout the following chapters.

FIGURE 2.6. The seed points. Then, we use the angles to form intersection points that will become new reference points.

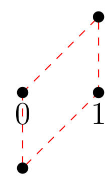


FIGURE 2.7. The first constructed points, $\llbracket 0,1 \rrbracket_{\frac{\pi}{4},\frac{\pi}{2}} = 1+i$ and $\llbracket 1,0 \rrbracket_{\frac{\pi}{4},\frac{\pi}{2}} = -i$.

Notice that this is all of the integral linear combinations of 1 and i, the imaginary unit. In fact, this is the construction that gives rise to the Gaussian integers, a known subring of the complex numbers.

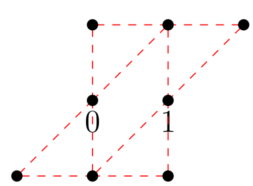


FIGURE 2.8. The last generation can be used as reference points, so $\llbracket 1+i,0 \rrbracket_{0,\frac{\pi}{2}} = i$, $\llbracket 1+i,1 \rrbracket_{0,\frac{\pi}{4}} = 2+i$, $\llbracket -i,0 \rrbracket_{0,\frac{\pi}{4}} = -1-i$, and $\llbracket -i,1 \rrbracket_{0,\frac{\pi}{2}} = 1-i$.

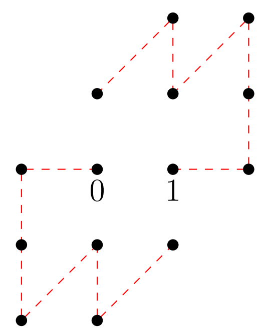


FIGURE 2.9. The previous generation can continue to produce points.

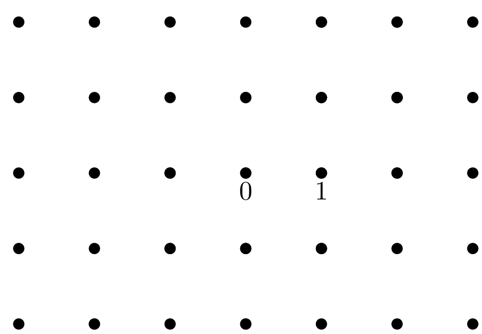


FIGURE 2.10. The origami set from seed points 0 and 1 with angles $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$.

However, $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ does not form a group, because $\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$, and $\frac{3\pi}{4}$ is not an angle used in the construction, so the set of angles is not closed under addition. So this origami ring, $\mathbb{Z}[i]$, is subject to some constraint other than the angle set that gives $\mathbb{Z}[i]$ its ring structure. We will work towards identifying it. Also notice that $\mathbb{Z}[i]$ has *topological* structure, forming a lattice rather than a dense set.

3. Origami Lattices

The constructions above create very beautifully ordered arrangement of regular points on $\mathbb{C}$. Surely, there is some type of way to characterize this phenomenon.

DEFINITION (Lattice). Let V be a vector space of dimension n over $\mathbb{R}$. A *lattice* Λ is a subgroup of the form $\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_r$, with $e_1, \ldots, e_r$ linearly independent elements of V [**Mil17**].

We can know that the Gaussian integers is a lattice, where the basis is given by $\{1, i\}$. We also know that our other example, made with $U = \{0, \frac{\pi}{3}, \frac{2\pi}{3}\}$ forms a lattice where the basis is given by $\{1, \frac{\sqrt{3}}{2}i\}$. When is an origami set a lattice?

THEOREM 2.4 ([**BR16**]). *Let $U = \{0, \alpha, \beta\}$. Then $M(U)$ is a lattice in $\mathbb{C}$ with the form $M(U) = \mathbb{Z} + \mathbb{Z}[\![0, 1]\!]_{\alpha,\beta}$.*

PROOF. From Theorem 2.3, we see that $M(U)$ is a subgroup of $\mathbb{C}$ under addition. Since $0 \in M(U)$, and $[\![0, 1]\!]_{\alpha,\beta} \in M(U)$, we can see that $\mathbb{Z}+\mathbb{Z}\cdot[\![0, 1]\!]_{\alpha,\beta} \subseteq M(U)$. We will prove that $M(U) \subseteq \mathbb{Z}+\mathbb{Z}\cdot[\![0, 1]\!]_{\alpha,\beta}$ by induction on the generations of the set. We know from Equation 2.4 that $M_1 = \{0, 1, [\![0, 1]\!]_{\alpha,\beta}, [\![1, 0]\!]_{\alpha,\beta} \subseteq \mathbb{Z} + \mathbb{Z} \cdot [\![0, 1]\!]_{\alpha,\beta}$, proving our base case that $M_1 \subseteq \mathbb{Z} + \mathbb{Z} \cdot [\![0, 1]\!]_{\alpha,\beta}$.

Our induction hypothesis is then that p and q are in $M_n \subseteq \mathbb{Z} + \mathbb{Z} \cdot [\![0, 1]\!]_{\alpha,\beta}$, for some $n \in \mathbb{N}$. Let $u, v \in U$.

We will show that $\llbracket p, q \rrbracket_{u,v} \in \mathbb{Z} + \mathbb{Z} \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}$. From Proposition 2.1, we know $\llbracket p, q \rrbracket_{u,v} = \llbracket p, 0 \rrbracket_{u,v} + \llbracket 0, q \rrbracket_{u,v}$, so we only need to show that

$$\llbracket a + b \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}, 0 \rrbracket_{u,v} \in \mathbb{Z} + \mathbb{Z} \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}.$$

Here, $a + b \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}$ is an element of M_n. By linearity, we know that

$$\llbracket a + b \cdot \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v} = \llbracket a, 0 \rrbracket_{u,v} + \llbracket b \cdot \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v}$$

$$\llbracket a + b \cdot \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v} = a \llbracket 1, 0 \rrbracket_{u,v} + \llbracket b \cdot \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v}$$

$$\llbracket a + b \cdot \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v} = a \llbracket 1, 0 \rrbracket_{u,v} + b \cdot \llbracket \llbracket 0, 1 \rrbracket_{u,v}, 0 \rrbracket_{u,v}.$$

First, we'll consider $a \llbracket 1, 0 \rrbracket_{u,v}$. Now, $\llbracket 1, 0 \rrbracket_{u,v}$ must be an element of M_1, so $\llbracket 1, 0 \rrbracket_{u,v} = 1,\ 0,\ \llbracket 0, 1 \rrbracket_{\alpha,\beta}$, or $\llbracket 1, 0 \rrbracket_{\alpha,\beta}$. If $u = 0$ or $v = 0$, then $\llbracket 1, 0 \rrbracket_{u,v} = 0$ or $\llbracket 1, 0 \rrbracket_{u,v} = 1$, which can be seen the following figure:

FIGURE 2.11. The seed points.

If $u, v \neq 0$, then either $u = \alpha, v = \beta$ or $v = \alpha, u = \beta$. If $u = \alpha, v = \beta$, then $\llbracket 1, 0 \rrbracket_{u,v} = \llbracket 1, 0 \rrbracket_{\alpha,\beta}$. On the other hand, $v = \alpha, u = \beta$, then $\llbracket 1, 0 \rrbracket_{u,v} = \llbracket 1, 0 \rrbracket_{\beta,\alpha} = \llbracket 0, 1 \rrbracket_{\alpha,\beta}$.

Next, consider $b \cdot \llbracket \llbracket 0, 1 \rrbracket_{\alpha,\beta}, 0 \rrbracket_{u,v}$. So, we just need to show that $\llbracket \llbracket 0, 1 \rrbracket_{\alpha,\beta}, 0 \rrbracket_{u,v} \in \mathbb{Z} + \mathbb{Z} \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}$. There are $\binom{3}{2} = 6$ possible cases here.

(1) $[(u, v) = (\alpha, \beta)]$ See that $\llbracket 1, 0 \rrbracket_{\alpha,\beta} = r\alpha$ for some $r \in \mathbb{R}$. Then, $\llbracket \llbracket 0, 1 \rrbracket_{\alpha,\beta}, 0 \rrbracket_{\alpha,\beta} = r \llbracket \alpha, 0 \rrbracket_{\alpha,\beta} = 0$. We know that $0 \in M_1$, and hence $0 \in \mathbb{Z} + \mathbb{Z} \cdot \llbracket 0, 1 \rrbracket_{\alpha,\beta}$.

(2) $[(u, v) = (\beta, \alpha)]$ We know that $\llbracket \llbracket 0, 1 \rrbracket_{\alpha,\beta}, 0 \rrbracket_{\beta,\alpha}$ is the projection of $\llbracket 0, 1 \rrbracket_{\alpha,\beta}$ onto the line $r\alpha$ in the direction of β. Further, we

know that $[\![0,1]\!]_{\alpha,\beta} \in \mathbb{R}\alpha$, so $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,\alpha} = [\![0,1]\!]_{\alpha,\beta}$, which we know to be in $\mathbb{Z} + \mathbb{Z} \cdot [\![0,1]\!]_{\alpha,\beta}$.

(3) $[(u,v) = (\alpha, 0)]$ We know that $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\alpha,0}$ is the projection of $[\![0,1]\!]_{\alpha,\beta}$ onto the real axis in the direction of α. So, $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\alpha,0} = 0$.

(4) $[(u,v) = (\beta, 0)]$ We know that $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,0}$ is the projection of $[\![0,1]\!]_{\alpha,\beta}$ onto $r\beta$ in the direction of the real axis. The line extending from 1 in the direction β intersects with $[\![0,1]\!]_{\alpha,\beta}$. So, $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,0} = 1$.

(5) $[(u,v) = (0, \alpha)]$ We know that $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\alpha}$ is the line crossing through $[\![0,1]\!]_{\alpha,\beta} + s$ and $r\alpha$ for $r, s \in \mathbb{R}$. Since $x \in \mathbb{R}\alpha$, then $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\alpha} = [\![0,1]\!]_{\alpha,\beta}$.

(6) $[(u,v) = (0, \alpha)]$ See that $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\beta} + [\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,0} = [\![0,1]\!]_{\alpha,\beta}$, while $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,0} = 1$. So,

$$[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\beta} = [\![0,1]\!]_{\alpha,\beta} - [\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{\beta,0}$$
$$[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\beta} = [\![0,1]\!]_{\alpha,\beta} - 1.$$

Hence, $[\![[\![0,1]\!]_{\alpha,\beta},0]\!]_{0,\beta} \in \mathbb{Z} + \mathbb{Z} \cdot [\![0,1]\!]_{\alpha,\beta}$.

We have shown that M_n, the $n-$th generation of M, is contained in $\mathbb{Z} + \mathbb{Z} \cdot [\![0,1]\!]_{\alpha,\beta}$. This completes our proof by induction, and thus, $M(U) = \mathbb{Z} + \mathbb{Z} \cdot [\![0,1]\!]_{\alpha,\beta}$ when $U = \{0, \alpha, \beta\}$.

$\square$

What does it mean for $M(U)$ to not be a lattice? We must first consider some definitions from topology. Topological space comes from constructing topology on a set.

Definition (Topology [**Mun75**]). A *topology* on a set X is a collection $\mathfrak{J}$ of subsets of X such that:

(1) Both $\emptyset$ and X are in $\mathfrak{J}$.
(2) The union of the elements of any subcollection of $\mathfrak{J}$ is in $\mathfrak{J}$.
(3) The intersection of the elements of any finite subcollection of $\mathfrak{J}$ is in $\mathfrak{J}$.

A set X for which a topology $\mathfrak{J}$ has been specified is called a *topological space*. A topological space is an ordered pair $(X, \mathfrak{J})$ consisting of a set X and a topology $\mathfrak{J}$ on X.

For a topological space X, given $a \in X$ and $\varepsilon < 0$, recall that the $\varepsilon-$neighborhood of a is the set $V_\varepsilon(a) = \{x \in X : |x - a| < \varepsilon\}$ [**Abb01**].

Definition (Limit Point). A point x is a *limit point* of a set A if every $\varepsilon-$neighborhood $V_\varepsilon(x)$ of x intersects the set A at some point other than x.

We can think of limit points as "cluster points," but if a point x is a limit point of a set A, then x is the limit of a sequence in A. Limit points can tell us about the "closeness" of points in a set.

Definition (Dense Set). Let X be a topological space. A subset A of X is *dense in* X if every point x in X either belongs to A or is a limit point of A.

In terms of familiar sets, if we consider $X = \mathbb{R}$, and $A = \mathbb{Q}$, we can see that every real number is either in $\mathbb{Q}$ or a limit point of $\mathbb{Q}$. We can also see a dense origami set. Using the same seed points as usual, $\{0, 1\}$ and angles from $\left\{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}\right\}$, we get the following construction. Let's denote this set of angles as $\tilde{U} = \left\{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}\right\}$.

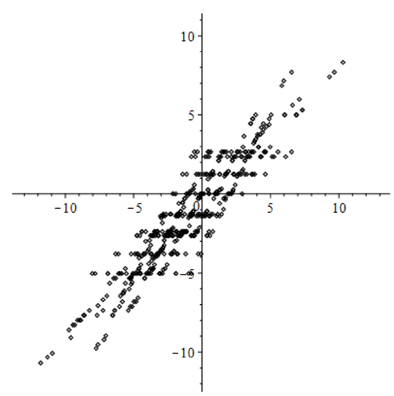


FIGURE 2.12. Three generations of the origami construction from $\{0, 1\}$ and $\tilde{U} = \left\{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}\right\}$ [**LLN+18**].

See that this construction involves four angles. Could the dense nature of the construction come from the fact that we have too many possibilities of intersections from each reference point? In fact, the answer is yes.

THEOREM 2.5 ([**BBDLG12**]). *If U has more than 3 angles, then $M(U)$ is dense in the complex plane.*

PROOF. Let n be the number of angles in U. If n is nonprime, then $\frac{1}{n}$ is in the ring. Then the origami ring has points arbitrarily close to zero, and therefore $M(U)$ is dense. Alternatively, if n is prime, then the image of the ring of integers under a complex embedding is dense if there is more than one pair of conjugate embeddings and this happens for prime n when $n > 3$ [**Mar77**]. $\square$

Now, a new question arises regarding the structure of an origami construction, $M(U)$. When does $M(U)$ have algebraic structure (i.e., is a subring of $\mathbb{C}$), and when does $M(U)$ have topological structure (i.e., is a lattice)? The two constructions that we did stepwise, to construct $\mathbb{Z}[\zeta_3]$ and $\mathbb{Z}[i]$, are both lattices; we know that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$, and by Theorem 2.3, $\mathbb{Z}[\zeta_3]$ is a ring. We have just noted that for the construction from the same seed points, but the angles from $\tilde{U}$ is not a lattice. We will eventually show that $M(\tilde{U})$ is a ring.

4. The Ring Structure of $M_{\mathbb{R}}$

Recall that $[\![r,s]\!]_{\alpha,\beta}$ refers to the intersection point of the extension from r along α and the extension from s along β. Additionally, we denote the $\delta-$projection for a point $[\![p,0]\!]_{\alpha,\beta}$ by $\delta\left([\![p,0]\!]_{\alpha,\beta}\right)$, to represent the intersection of the real axis with the line of slope δ through $[\![p,0]\!]_{\alpha,\beta}$.

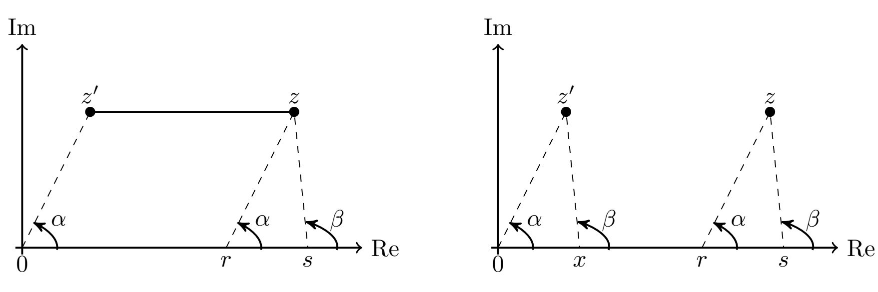
Lemma 2.6 ([**Möl18**]). *Let $\alpha, \beta \in (0,\pi)$ be two different angles. For all $r, s \in \mathbb{R}$, the two equations: $\alpha\left([\![r,s]\!]_{\alpha,\beta}\right) = r$, and $\beta\left([\![r,s]\!]_{\alpha,\beta}\right) = s$ hold. Additionally, for all $z \in \mathbb{C}$, $z = [\![\alpha(z), \beta(z)]\!]_{\alpha,\beta}$.*

Proof. The proof of Lemma 2.6 can be written using Equation 2.4, again. We know $[\![r,s]\!]_{\alpha,\beta}$ is a projection of r in the direction of α and a projection of s in the direction of β by Proposition 2.1.3. Hence, the projection of $[\![r,s]\!]_{\alpha,\beta}$ in the direction of α onto the real axis, or $\alpha\left([\![r,s]\!]_{\alpha,\beta}\right)$ is in fact r. Likewise, the projection of $[\![r,s]\!]_{\alpha,\beta}$ in the direction of β onto the real axis, $\beta\left([\![r,s]\!]_{\alpha,\beta}\right) = s$. $\square$

Let's denote the real part of an origami set $M(U)$ as $M_{\mathbb{R}}$, that is, $M_{\mathbb{R}} = M(U) \cap \mathbb{R}$. We will show that the structure of $M_{\mathbb{R}}$ can give us information about the structure of $M(U)$. We will first show that $M_{\mathbb{R}}$ is a subgroup of $\mathbb{R}$. Consider z, an element of M. The projections $\alpha(z)$ and $\beta(z)$ are elements of $M_{\mathbb{R}}$. Additionally, see that Lemma 2.6 shows that $\alpha(z) = \beta(z) = z$ for all $z \in M_{\mathbb{R}}$. If r, s are elements of $M_{\mathbb{R}}$, then $[\![r, s]\!]_{\alpha,\beta}$ is an element of M, and the $\alpha-$ and $\beta-$projections of $[\![r, s]\!]_{\alpha,\beta}$ are elements of $M_{\mathbb{R}}$.

LEMMA 2.7. *$M_{\mathbb{R}}$ is an additive subgroup of $\mathbb{R}$.*

PROOF. This utilizes the one-step subgroup test. Let r, s be elements of the non-empty set $M_{\mathbb{R}}$, and let $s \geq r$, and we will show that both $s-r$ and $r-s$ are in $M_{\mathbb{R}}$. We'll define $z := [\![r, s]\!]_{\alpha,\beta}$, and note that the so-called "$(\alpha, \beta)-$ coordinates" of z are given by (r, s). By the definition of an origami set M, the intersection point point z' of the lines $z+\mathbb{R}e^{i0}$ and $0+\mathbb{R}e^{i\alpha}$ is an element of M. We know that z' is the projection of $(0, x)$, for an appropriate $x \in M_{\mathbb{R}}$, as seen in the following figure.

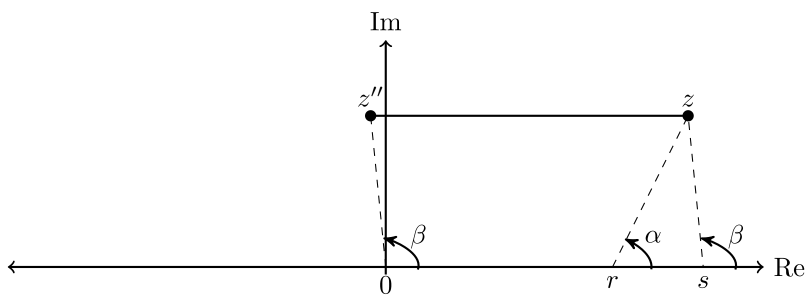


(a) z and extension to z' (b) z' is an origami point

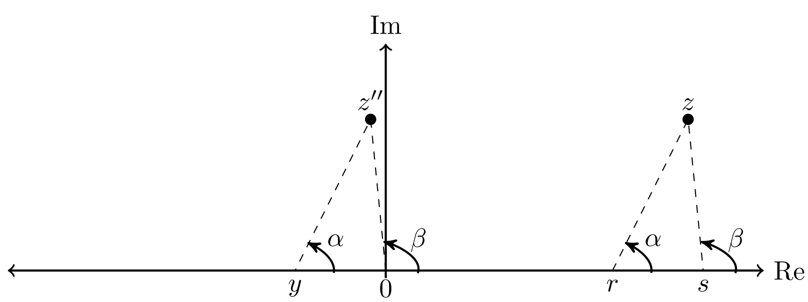
FIGURE 2.13. The construction of z'.

See that we have a triangle with vertices r, z, s that is congruent to the triangle with vertices 0, z', x. Corresponding sides of the triangles have the same length, so $s - r = x - 0 = x$. We know that x must be in $M_{\mathbb{R}}$, implying that $s - r \in M_{\mathbb{R}}$.

Now, consider the point z'' defined by the intersection of $z + \mathbb{R}e^{i0}$ and $0 + \mathbb{R}e^{i\beta}$. In an extremely similar fashion to the process above, we will show that $r - s \in M_{\mathbb{R}}$.



(a) z and extension to z''



(b) z'' is an origami point

FIGURE 2.14. The construction of z''.

See that we have a triangle with vertices r, z, s that is congruent to the triangle with vertices y, z'', 0. Corresponding sides of the triangles have the same length, so $r - s = y - 0 = y$. We know that y must be in $M_{\mathbb{R}}$, implying that $r - s \in M_{\mathbb{R}}$.

We have shown that $M_{\mathbb{R}}$ is an additive subgroup of $\mathbb{R}$.

□

Note that since $0, 1 \in M_{\mathbb{R}}$, it follows that $\mathbb{Z} \subseteq M_{\mathbb{R}}$. From Lemma 2.7, we can find an explicit description of origami sets, and use it to show that M is an additive subgroup of $\mathbb{C}$.

Theorem 2.8. *The origami set M is the $M_{\mathbb{R}}$−span of 1 and $[\![0,1]\!]_{\alpha,\beta}$:*

$$M = M_{\mathbb{R}} + M_{\mathbb{R}} \cdot [\![0,1]\!]_{\alpha,\beta} = \{r + s \cdot [\![0,1]\!]_{\alpha,\beta} : r, s \in M_{\mathbb{R}}\}.$$

Proof. Let z be an element of M. Then there exist some $r, s \in M_{\mathbb{R}}$ for which $z = [\![r,s]\!]_{\alpha,\beta}$. From Proposition 2.1, we know:

$$\begin{aligned}
z &= [\![r,s]\!]_{\alpha,\beta} \\
z &= [\![r,0]\!]_{\alpha,\beta} + [\![0,s]\!]_{\alpha,\beta} \\
z &= r \cdot [\![1,0]\!]_{\alpha,\beta} + s \cdot [\![0,1]\!]_{\alpha,\beta} \\
z &= r \cdot (1 - [\![0,1]\!]_{\alpha,\beta}) + s \cdot [\![0,1]\!]_{\alpha,\beta} \\
z &= r \cdot 1 - r \cdot [\![0,1]\!]_{\alpha,\beta} + s \cdot [\![0,1]\!]_{\alpha,\beta} \\
z &= r \cdot 1 + (-r + s) \cdot [\![0,1]\!]_{\alpha,\beta}.
\end{aligned}$$

Since $r, s \in M_{\mathbb{R}}$, then $r, s \in \mathbb{R}$, an abelian group, so $-r + s = s - r$. From Lemma 2.7, we know $s - r \in M_{\mathbb{R}}$, so $z \in M_{\mathbb{R}} + M_{\mathbb{R}} \cdot [\![0,1]\!]_{\alpha,\beta}$.

Now, assume that $r, s \in M_{\mathbb{R}}$, and we will show that $r + s \cdot [\![0,1]\!]_{\alpha,\beta} \in M$. Starting with $r + s \cdot [\![0,1]\!]_{\alpha,\beta}$, by Proposition 2.1,

$$r + s \cdot [\![0,1]\!]_{\alpha,\beta} = [\![r,r]\!]_{\alpha,\beta} + s \cdot [\![0,1]\!]_{\alpha,\beta}$$

$$r + s \cdot [\![0,1]\!]_{\alpha,\beta} = [\![r,r]\!]_{\alpha,\beta} + [\![0,s]\!]_{\alpha,\beta}$$

$$r + s \cdot [\![0,1]\!]_{\alpha,\beta} = [\![r,r+s]\!]_{\alpha,\beta}.$$

From Lemma 2.7, we know that $r + s \in M_{\mathbb{R}}$, which implies that $[\![r,r+s]\!]_{\alpha,\beta}$, and therefore, $r + s \cdot [\![0,1]\!]_{\alpha,\beta}$ is in fact an element of M. So, we can write $M = \{r + s \cdot [\![0,1]\!]_{\alpha,\beta} : r, s \in M_{\mathbb{R}}\}$.

□

Our next goal is to show that $M_{\mathbb{R}}$ is a *subring* of $\mathbb{R}$, mainly using coordinate transformations. From our set of angles U, we choose any two angles γ and δ. Then, we can convert $(\alpha, \beta)-$coordinates into $(\gamma, \delta)-$coordinates.

Definition ($\gamma-$projection). Let $\gamma \in U \setminus \{0\}$ be arbitrary. Denote the $\gamma-$*projection* of $[\![0,1]\!]_{\alpha,\beta}$ with

$$p(\gamma) := \gamma\left([\![0,1]\!]_{\alpha,\beta}\right).$$

See that $p(\gamma) \in M_{\mathbb{R}}$. Lemma 2.6 tells us $p(\alpha) = 0$, and $p(\beta) = 1$. Additionally, if $\gamma \neq \delta$, then $p(\gamma) \neq p(\delta)$. So, there is a one-to-one map $p : U \setminus \{0\} \to M_{\mathbb{R}}$. The following figure shows the coordinate conversion of an origami point through projections.

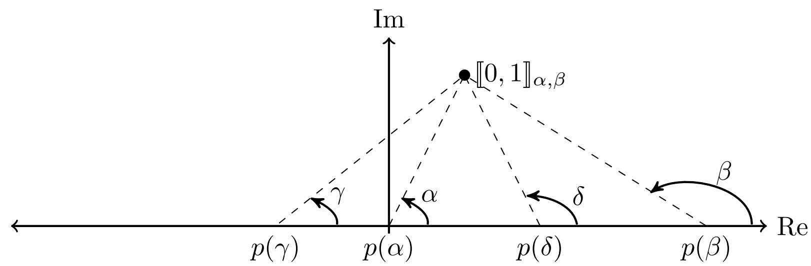


FIGURE 2.15. Different projections of $[\![0,1]\!]_{\alpha,\beta}$.

In Figure 4.15, note that for $[\![0,1]\!]_{\alpha,\beta}$, $p(\alpha) = 0$, and $p(\beta) = 1$. When transitioning to different pairs of angles, we can show that the difference between our new projection points are still in $M_{\mathbb{R}}$.

PROPOSITION 2.9 (Coordinate Conversion). *Let $\gamma, \delta \in U \setminus \{0\}$ be two different angles. For any $r, s \in M_{\mathbb{R}}$, the following equations hold.*

(1) $[\![r,s]\!]_{\alpha,\beta} = [\![r + (s-r)p(\gamma), r + (s-r)p(\delta)]\!]_{\gamma,\delta}$
(2) $[\![r,s]\!]_{\alpha,\beta} = \left[\!\!\left[\dfrac{sp(\gamma) - rp(\delta)}{p(\gamma) - p(\delta)}, \dfrac{r - s + sp(\gamma) - rp(\delta)}{p(\gamma) - p(\delta)}\right]\!\!\right]_{\alpha,\beta}$.

PROOF. This proof utilizes the properties of Proposition 2.1 and previous lemmas.

(1) By the definition of a projection, we know $p(\gamma) = \gamma\left([\![0,1]\!]_{\alpha,\beta}\right)$, and $p(\delta) = \delta\left([\![0,1]\!]_{\alpha,\beta}\right)$. So, $[\![0,1]\!]_{\alpha,\beta} = [\![p(\gamma), p(\delta)]\!]_{\gamma,\delta}$. Then, by Lemma 2.6, we can see that:

$$\begin{aligned}
[\![1,0]\!]_{\alpha,\beta} &= 1 - [\![0,1]\!]_{\alpha,\beta} \\
&= [\![1,1]\!]_{\alpha,\beta} - [\![0,1]\!]_{\alpha,\beta} \\
&= [\![1,1]\!]_{\alpha,\beta} - [\![p(\gamma), p(\delta)]\!]_{\gamma,\delta} \\
&= [\![1,1]\!]_{\gamma,\delta} - [\![p(\gamma), p(\delta)]\!]_{\gamma,\delta} \\
&= [\![1 - p(\gamma), 1 - p(\delta)]\!]_{\gamma,\delta}.
\end{aligned}$$

Now, we can represent $[\![r,s]\!]_{\alpha,\beta}$ in $(\gamma,\delta)\,-$coordinates.

(2) If $[\![r,s]\!]_{\gamma,\delta} = [\![x,y]\!]_{\alpha,\beta}$ for some $x, y \in M_{\mathbb{R}}$, then by Proposition 2.8.1, x and y satisfy the following linear system:

$$\begin{Bmatrix} x + (y-x)p(\gamma) & = & r \\ x + (y-x)p(\delta) & = & s \end{Bmatrix}.$$

Since we have chosen $\gamma \neq \delta$, then we know that $p(\gamma) \neq p(\delta)$, so there is a unique solution (x, y) to the system. Then, since this is a linear system, we may row reduce the appropriate augmented matrix to find the solution:

$$\operatorname{rref}\begin{bmatrix} 1-p(\gamma) & p(\gamma) & r \\ 1-p(\delta) & p(\delta) & s \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dfrac{sp(\gamma)-rp(\delta)}{p(\gamma)-p(\delta)} \\ 0 & 1 & \dfrac{r-s+sp(\gamma)-rp(\delta)}{p(\gamma)-p(\delta)} \end{bmatrix}.$$

Hence we have shown that

$$[\![r,s]\!]_{\alpha,\beta} = \left[\!\!\left[\frac{sp(\gamma)-rp(\delta)}{p(\gamma)-p(\delta)}, \frac{r-s+sp(\gamma)-rp(\delta)}{p(\gamma)-p(\delta)} \right]\!\!\right]_{\alpha,\beta},$$

completing our proof.

□

Moving forward, we consider the differences $p(\gamma) - p(\delta)$, and the quotients $(p(\gamma) - p(\delta))^{-1}$, and show that both quantities are elements of $M_{\mathbb{R}}$. We introduce some new notation here.

Definition (Δ). We denote the set of all differences $p(\gamma) - p(\delta)$ by Δ, where $\gamma \neq \delta$, and $\gamma, \delta \in U \setminus \{0\}$:

$$\Delta := \{p(\gamma) - p(\delta) : \gamma, \delta \in U \setminus \{0\} \text{ and } \gamma \neq \delta\}.$$

Note, in particular, that 0 is *not* an element of Δ. Hence, we define the set Δ^{-1}.

Definition (Δ^{-1}). The reciprocals of differences $p(\gamma) - p(\delta)$ can be a different set, denoted by Δ^{-1} :

$$\Delta^{-1} := \{d^{-1} : d \in \Delta\}$$
$$\Delta^{-1} = \left\{ \frac{1}{p(\gamma) - p(\delta)} : \gamma, \delta \in U \setminus \{0\} \text{ and } \gamma \neq \delta \right\}.$$

See that Δ generates a subring of $\mathbb{R}$, we will denote this subring as $\mathbb{Z}[\Delta]$.

Lemma 2.10. *The subring $\mathbb{Z}[\Delta]$ is a subset of $M_{\mathbb{R}}$.*

Proof. First, let R be the ring $\mathbb{Z}[p(\gamma) : \gamma \in U \setminus \{0\}]$. We will show that $R = \mathbb{Z}[\Delta]$ by showing that the two sets are subsets of each other. Clearly, $\mathbb{Z}[\Delta] \subseteq R$. To show $R \subseteq \mathbb{Z}[\Delta]$, we consider some element $r \in R$. By the way that we defined R, r is just a sum of addents: $z \cdot p(\gamma_1) \cdots p(\gamma_s)$, where $z \in \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_s \in U \setminus \{0\}$. Now, since $p(\alpha) = 0$, then:

$$r = z \cdot p(\gamma_1) \cdots p(\gamma_s) = z \cdot (p(\gamma_1) - p(\alpha)) \cdots (p(\gamma_s) - p(\alpha)).$$

Notice that $z \cdot (p(\gamma_1) - p(\alpha)) \cdots (p(\gamma_s) - p(\alpha)) \in \mathbb{Z}[\Delta]$, so $r \in \mathbb{Z}[\Delta]$. Now, we'll show that $p(\gamma) \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$ for all $\gamma \in U \setminus \{0\}$. Choose an arbitrary $\gamma \in U \setminus \{0\}$ and $s \in M_{\mathbb{R}}$. Then, $[\![0, s]\!]_{\alpha,\beta} \in M$, and so $\gamma([\![0, s]\!]_{\alpha,\beta}) \in M_{\mathbb{R}}$. By linearity, $\gamma([\![0, s]\!]_{\alpha,\beta}) = p(\gamma) \cdot s$. Since $\mathbb{Z}$ is a subset of $M_{\mathbb{R}}$, repeatedly applying the previous result shows that $M_{\mathbb{R}}$ contains all of the products $z \cdot p(\gamma_1) \cdots p(\gamma_s)$, where $s \in \mathbb{N}$, $z \in \mathbb{Z}$, and $\gamma_1, \ldots, \gamma_s \in U \setminus \{0\}$. Then, since $M_{\mathbb{R}}$ is closed under addition, $R \subseteq M_{\mathbb{R}}$, and hence $\mathbb{Z}[\Delta] \subseteq M_{\mathbb{R}}$, our desired result. $\square$

Notice that $\Delta \cup \Delta^{-1}$ also generates a subring of $\mathbb{R}$. Let's denote this subring as $\mathbb{Z}[\Delta, \Delta^{-1}]$.

LEMMA 2.11. *The subring* $\mathbb{Z}[\Delta, \Delta^{-1}]$ *is a subset of* $M_{\mathbb{R}}$.

PROOF. Consider $(p(\gamma) - p(\delta))^{-1} \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$ for all $\gamma, \delta \in U \setminus \{0\}$, where $\gamma \neq \delta$. First, we'll show that $p(\gamma)^{-1} \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$ is true for all $\gamma \in U \setminus \{0, \alpha\}$. Let γ and $r \in M_{\mathbb{R}}$ be arbitrary elements. Then, $[\![r, 0]\!]_{\gamma,\alpha} \in M$, so $\beta([\![r, 0]\!]_{\gamma,\alpha}) \in M_{\mathbb{R}}$. Recall that $p(\alpha) = 0$ because we have $[\![r, 0]\!]_{\gamma,\alpha}$. By Proposition 2.9,

$$\begin{aligned} \beta([\![r, 0]\!]_{\gamma,\alpha}) &= \frac{r - rp(\alpha)}{p(\gamma) - p(\alpha)} \\ &= \frac{r}{p(\gamma)}. \end{aligned}$$

Hence, $\dfrac{r}{p(\gamma)} \in M_{\mathbb{R}}$, so we have shown that $p(\gamma)^{-1} \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$. Next, we'll show that $(p(\gamma) - p(\delta))^{-1} \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$. Consider some $\gamma, \delta \in U \setminus \{0\}$, where $\gamma \neq \delta$, and $s \in M_{\mathbb{R}}$. Suppose $\gamma \neq \alpha$. Then, $p(\gamma)^{-1}s \in M_{\mathbb{R}}$, so $[\![0, p(\gamma)^{-1}s]\!]_{\gamma,\delta} \in M$. We know that $\alpha([\![0, p(\gamma)^{-1}s]\!]_{\gamma,\delta}) \in M_{\mathbb{R}}$. By Proposition 2.9,

$$\begin{aligned} \alpha\left([\![0, p(\gamma)^{-1}s]\!]_{\gamma,\delta}\right) &= \frac{p(\gamma)^{-1}s \cdot p(\gamma)}{p(\gamma) - p(\delta)} \\ &= \frac{s}{p(\gamma) - p(\delta)}. \end{aligned}$$

Hence, $\dfrac{s}{p(\gamma) - p(\delta)} \in M_{\mathbb{R}}$, and we have shown $(p(\gamma) - p(\delta))^{-1} \cdot M_{\mathbb{R}} \subseteq M_{\mathbb{R}}$. Note that if $\gamma = \alpha$, then $\delta \neq \alpha$. □

We'll use $\mathbb{Z}[\Delta, \Delta^{-1}]$ to describe $M_{\mathbb{R}}$, and identify some more of the properties of $M_{\mathbb{R}}$.

THEOREM 2.12. *The equality* $M_{\mathbb{R}} = \mathbb{Z}[\Delta, \Delta^{-1}]$ *holds, showing that* $M_{\mathbb{R}}$ *is a subring of* $\mathbb{R}$.

PROOF. From our work in Lemma 2.11, we have already shown that $\mathbb{Z}[\Delta, \Delta^{-1}] \subseteq M_{\mathbb{R}}$, so the only step left to show for set equality is that $M_{\mathbb{R}} \subseteq \mathbb{Z}[\Delta, \Delta^{-1}]$. The subring fact follows because we already know that $\mathbb{Z}[\Delta, \Delta^{-1}]$ is a subring of $\mathbb{R}$. To show that $M_{\mathbb{R}} \subseteq \mathbb{Z}[\Delta, \Delta^{-1}]$, we proceed through a proof by induction on projections of generations of the origami set; in other words, that $\alpha(M_k)$ and $\beta(M_k)$ are subsets of $\mathbb{Z}[\Delta, \Delta^{-1}]$. Recall that M_k is a generation of the origami set, and each is defined recursively, where

$$M = \bigcup_{k=0}^{\infty} M_k.$$

For $k = 0$, $\alpha(M_0) = \beta(M_0) = M_0 = \{0, 1\} \subseteq \mathbb{Z}[\Delta, \Delta^{-1}]$. Now, let's assume our induction hypothesis, that for some $k \in \mathbb{N}$, both $\alpha(M_k) \subseteq \mathbb{Z}[\Delta, \Delta^{-1}]$ and $\beta(M_k) \subseteq \mathbb{Z}[\Delta, \Delta^{-1}]$. We'll show that our claim holds for $k+1$. Let z be some element of M_{k+1}. We know that there exist $x, y \in M_k$, and angles $\gamma, \delta \in U$, where $\gamma \neq \delta$ such that

$$z = [\![x, y]\!]_{\gamma,\delta}.$$

First, we'll assume that $\gamma \neq 0$ and $\delta \neq 0$. Then, by Proposition 2.9,

$$\gamma(z) = \alpha(x) + (\beta(x) - \alpha(x))\, p(\gamma).$$

We know $\alpha(x)$ and $\beta(x)$ are elements of $\mathbb{Z}[\Delta, \Delta^{-1}]$. By the induction hypothesis, $\gamma(x) \in \mathbb{Z}[\Delta, \Delta^{-1}]$, and $\delta(y) \in \mathbb{Z}[\Delta, \Delta^{-1}]$. Now, z has $(\gamma, \delta)-$coordinates $(\gamma(x), \delta(y))$, so by Proposition 2.9,

$$\begin{aligned}
[\![\alpha(z), \beta(z)]\!]_{\alpha,\beta} &= z \\
&= [\![\gamma(x), \delta(y)]\!]_{\gamma,\delta} \\
&= \left[\!\!\left[\frac{\delta(y)p(\gamma) - \gamma(x)p(\delta)}{p(\gamma) - p(\delta)}, \frac{\gamma(x) - \delta(x) + \delta(y)p(\gamma) - \gamma(x)p(\delta)}{p(\gamma) - p(\delta)} \right]\!\!\right]_{\alpha,\beta}.
\end{aligned}$$

Hence, $\alpha(z), \beta(z) \in \mathbb{Z}[\Delta, \Delta^{-1}]$.

Next, let's assume that $\gamma = 0$. Note that this implies $\delta \neq 0$, and let's further assume that $\delta \neq \alpha$. Consider $(\alpha, \delta)-$coordinates. Then in a similar process as above, we can see that $\delta(x), \delta(y) \in \mathbb{Z}[\Delta, \Delta^{-1}]$. To show that $\alpha(z) \in \mathbb{Z}[\Delta, \Delta^{-1}]$, we'll return to our geometric thinking. Consider the line $x + \mathbb{R}$, seen in the following figure.

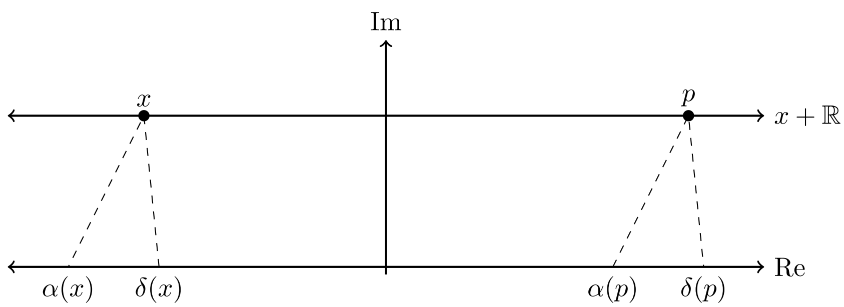


FIGURE 2.16. The point p lies on $x + \mathbb{R}$ if and only if $\alpha(p) - \delta(p) = \alpha(x) - \delta(x)$.

Note that $p \in \mathbb{C}$ must follow that condition because of the congruency of the triangles with vertices $\{\alpha(x), x, \delta(x)\}$ and $\{\alpha(p), p, \delta(p)\}$. So, by induction, $\alpha(x) \in \mathbb{Z}[\Delta, \Delta^{-1}]$. Hence, $\alpha(x) - \delta(x) \in \mathbb{Z}[\Delta, \Delta^{-1}]$. By the definition of z, we know that $\delta(z) = \delta(y)$. Since $z \in x + \mathbb{R}$, we know

$$\alpha(z) = \delta(y) + (\alpha(x) - \delta(x)),$$

and both $\delta(y), (\alpha(x) - \delta(x)) \in \mathbb{Z}[\Delta, \Delta^{-1}]$. Next, transforming the $(\alpha, \delta)-$coordinates of z using Proposition 2.9, we have

$$\beta(z) = \frac{\alpha(z) - \delta(y) - \alpha(z)p(\delta)}{-p(\delta)} \in \mathbb{Z}\left[\Delta, \Delta^{-1}\right].$$

The case $\delta = 0$ can be reduced to the same case as $\gamma = 0$. So, we have shown that $\alpha(z)$ and $\beta(z)$ for $z \in M_{k+1}$ are both elements of $\mathbb{Z}[\Delta, \Delta^{-1}]$, completing our induction. So, $M_{\mathbb{R}} = \mathbb{Z}[\Delta, \Delta^{-1}]$. $\square$

We have now shown that $M_{\mathbb{R}}$ is a subring of $\mathbb{R}$. Since we have shown that M is the $M_{\mathbb{R}}-$span of 1 and $[\![0,1]\!]_{\alpha,\beta}$, we can soon show when M is an origami ring.

5. Origami Rings

We can now provide criteria for an origami set to be an origami ring, beginning with some trigonometry, and we reach five equivalent statements. From there, we develop some corollaries. Finally, we show that $M(\tilde{U})$, our previous dense origami set, is an origami ring using our criteria.

Lemma 2.13. *We have some preliminaries about an origami set.*

(1) *The two equalities*

$$\begin{aligned} [\![0,1]\!]_{\alpha,\beta} &= -\frac{\cos\alpha\cdot\sin\beta}{\sin(\alpha-\beta)} - i\cdot\frac{\sin\alpha\cdot\sin\beta}{\sin(\alpha-\beta)} \\ |[\![0,1]\!]_{\alpha,\beta}|^2 &= \frac{\sin^2\beta}{\sin^2(\alpha-\beta)} \end{aligned}$$

hold.

(2) *The equation* $[\![0,1]\!]_{\alpha,\beta}\cdot[\![1,0]\!]_{\alpha,\beta} = \left[\!\!\left[\frac{\sin^2\beta}{\sin^2(\alpha-\beta)},\frac{\sin^2\alpha}{\sin^2(\alpha-\beta)}\right]\!\!\right]_{\alpha,\beta}$ *holds.*

(3) *For any* $\gamma\in U\setminus\{0\}$, *we have* $p(\gamma)=\frac{\sin(\alpha-\gamma)\sin\beta}{\sin(\alpha-\beta)\sin\gamma}$.

Proof. As a note, all of the quotients are well-defined because $\alpha,\beta,\gamma\in(0,\pi)$ and $\alpha\neq\beta$.

(1) By definition, $\{[\![0,1]\!]_{\alpha,\beta}\} = (0+\mathbb{R}e^{i\alpha})\cap\left(1+\mathbb{R}e^{i\beta}\right)$. Then, using our various representations for complex numbers, we see

that we must solve the equation $\lambda e^{i\alpha} = 1 + \mu e^{i\beta}$.

$$\begin{aligned} \lambda e^{i\alpha} &= 1 + \mu e^{i\beta} \\ \lambda\left(\cos\alpha + i\sin\alpha\right) &= 1 + \mu\left(\cos\beta + i\sin\beta\right) \end{aligned}$$

The real parts must be equal to each other, and the imaginary components must be equal to one another, and so we have:

$$\left\{\begin{array}{ccc} \lambda\cos\alpha & = & 1 + \mu\cos\beta \\ \lambda\sin\alpha & = & \mu\sin\beta \end{array}\right\}.$$

From here, solving for λ and μ gets us to the correct expression. This makes use of many trigonometric identities.

$$\begin{aligned} \lambda\cos\alpha &= 1 + \mu\cos\beta \\ \mu &= \frac{\lambda\cos\alpha - 1}{\cos\beta} \\ \lambda\sin\alpha &= \frac{\lambda\cos\alpha - 1}{\cos\beta}\sin\beta \\ \lambda\sin\alpha &= \frac{\lambda\cos\alpha\sin\beta - \sin\beta}{\cos\beta} \\ \lambda\sin\alpha &= \lambda\cos\alpha\tan\beta - \tan\beta \\ \tan\beta &= \lambda\cos\alpha\tan\beta - \lambda\sin\alpha \\ \lambda &= \frac{\tan\beta}{\cos\alpha\tan\beta - \sin\alpha} \\ \lambda &= -\frac{\tan\beta}{\sin\alpha - \cos\alpha\tan\beta} \\ \lambda &= -\frac{\sin\beta}{\sin(\alpha - \beta)} \end{aligned}$$

We substitute our new expression for λ into our expression for μ.

$$\mu = \frac{\lambda\cos\alpha - 1}{\cos\beta}$$
$$\mu = \frac{1}{\cos\beta}\left(-\sin\beta\frac{1}{\sin(\alpha-\beta)}\cos\alpha - 1\right)$$
$$\mu = -\tan\beta\frac{\cos\alpha}{\sin(\alpha-\beta)} - \frac{1}{\cos\beta}$$
$$\mu = -\frac{\sin\alpha}{\sin(\alpha-\beta)}.$$

Finally, combining these results for λ and μ with the expressions for the real and imaginary components, we have:

$$\{[\![0,1]\!]_{\alpha,\beta}\} = \lambda\cos\alpha + i\mu\sin\beta$$
$$\{[\![0,1]\!]_{\alpha,\beta}\} = -\frac{\cos\alpha\cdot\sin\beta}{\sin(\alpha-\beta)} - i\cdot\frac{\sin\alpha\cdot\sin\beta}{\sin(\alpha-\beta)}.$$

Squaring this result gives us the expression for $\left|[\![0,1]\!]_{\alpha,\beta}\right|^2$.

(2) To show the second part, we start with our last result. Computing z, we have

$$z := [\![0,1]\!]_{\alpha,\beta}\cdot[\![1,0]\!]_{\alpha,\beta}$$
$$z = [\![0,1]\!]_{\alpha,\beta}\cdot\left(1 - [\![0,1]\!]_{\alpha,\beta}\right).$$

The intersection $\mathbb{R}\cap(z+\mathbb{R}e^{i\alpha})$ gives the $\alpha-$projection of z, and the intersection $\mathbb{R}\cap\left(z+\mathbb{R}e^{i\beta}\right)$ gives the $\beta-$projection of z. By a similar trigonometric argument, we reach that

$$[\![0,1]\!]_{\alpha,\beta}\cdot[\![1,0]\!]_{\alpha,\beta} = \left[\!\!\left[\frac{\sin^2\beta}{\sin^2(\alpha-\beta)}, \frac{\sin^2\alpha}{\sin^2(\alpha-\beta)}\right]\!\!\right]_{\alpha,\beta}.$$

(3) The definition of $p(\gamma)$ tells us that $\{p(\gamma)\} = \mathbb{R} \cap (\llbracket 0,1 \rrbracket_{\alpha,\beta} + \mathbb{R}e^{i\gamma})$. Using Part (1) of this lemma, we set up the following system:

$$\left\{\begin{array}{ccc} \lambda\cos\alpha & = & -\dfrac{\cos\alpha\sin\beta}{\sin(\alpha-\beta)} + \mu\cos\gamma \\ 0 & = & -\dfrac{\sin\alpha\sin\beta}{\sin(\alpha-\beta)} + \mu\sin\gamma. \end{array}\right\}.$$

Solving it, in a similar manner as in Part (1) of this lemma and facts about projections, give us $p(\gamma) = \dfrac{\sin(\alpha-\gamma)\sin\beta}{\sin(\alpha-\beta)\sin\gamma}$, our desired result.

□

We are now ready for the main theorem of Möller's work [**Möl18**]. With Lemma 2.13, we are able to prove some of the criteria easily.

THEOREM 2.14. *For an origami set M, the following statements are equivalent:*

(1) *M is an origami ring.*
(2) *The complex number $\llbracket 0,1 \rrbracket_{\alpha,\beta}$ is integral over $M_{\mathbb{R}}$ of degree two (there exists a monic irreducible quadratic polynomial $f \in M_{\mathbb{R}}[X]$ such that $f(\llbracket 0,1 \rrbracket_{\alpha,\beta}) = 0$).*
(3) *Both $\dfrac{\sin^2\beta}{\sin^2(\alpha-\beta)}$ and $2\dfrac{\cos\alpha\sin\beta}{\sin(\alpha-\beta)}$ are elements of $M_{\mathbb{R}}$.*
(4) *Both $\dfrac{\sin^2\alpha}{\sin^2(\alpha-\beta)}$ and $\dfrac{\sin^2\beta}{\sin^2(\alpha-\beta)}$ are elements of $M_{\mathbb{R}}$.*
(5) *$\llbracket 0,1 \rrbracket_{\alpha,\beta} \cdot \llbracket 1,0 \rrbracket_{\alpha,\beta}$ is an element of M.*

PROOF. For readability, let $e := \llbracket 0,1 \rrbracket_{\alpha,\beta}$. Then, $1 - e = \llbracket 1,0 \rrbracket_{\alpha,\beta}$. We will show that each statement implies the next, and the final implies the first.

First, let's show that $(1 \Rightarrow 2)$. Let's assume that M is an origami ring. Then we know that $e^2 \in M$. By Theorem 2.8, there exist $r, s \in$

$M_{\mathbb{R}}$. where $e^2r + se$. So, the monic quadratic polynomial $f := X^2 - sX - r \in M_{\mathbb{R}}[X]$ has a zero of e. Since e is not a real number, f is irreducible, proving Theorem 2.14.2.

Now, we'll show that $(2 \Rightarrow 3)$. Let's assume that there exists a monic irreducible quadratic polynomial $f \in M_{\mathbb{R}}[X]$ such that $f([\![0,1]\!]_{\alpha,\beta}) = 0$. Since f is a real polynomial, the complex conjugate $\overline{e}$ of e is also a zero of f. Then,

$$f = (X - e)(X - \overline{e})$$
$$f = X^2 - 2\mathrm{Re}(e)X + |e|^2.$$

We know f to be in $M_{\mathbb{R}}[X]$, so $2\mathrm{Re}(e)$ and $|e|^2$ are elements of $M_{\mathbb{R}}$. Then, using the criteria of Lemma 2.13.1, we have the result of Theorem 2.14.3.

Next, we will show that $(3 \Rightarrow 4)$. Assume that both $\frac{\sin^2\beta}{\sin^2(\alpha-\beta)}$ and $2 \cdot \frac{\cos\alpha\sin\beta}{sin(\alpha-\beta)}$ are elements of $M_{\mathbb{R}}$. Using the angle difference identities,

$$\frac{\sin^2\alpha}{\sin^2(\alpha-\beta)} = 1 + 2\frac{\cos\alpha\sin\beta}{\sin(\alpha-\beta)} + \frac{\sin^2\beta}{\sin^2(\alpha-\beta)}.$$

Since $M_{\mathbb{R}}$ is closed under addition, the above equation shows that $\dfrac{\sin^2\alpha}{\sin^2(\alpha-\beta)} \in M_{\mathbb{R}}$, hence proving Theorem 2.14.4.

For our penultimate step, we will show that $(4 \Rightarrow 5)$. Let's assume that both $\dfrac{sin^2\alpha}{sin^2(\alpha-\beta)}$ and $\dfrac{sin^2\beta}{sin^2(\alpha-\beta)}$ are elements of $M_{\mathbb{R}}$. Consider the $(\alpha,\beta)-$ coordinates of $e(1-e)$. By Lemma 2.13.2, they are elements of $M_{\mathbb{R}}$, showing that $e(1-e) \in M$, the result of Theorem 2.14.5.

Finally, we'll show that $(5 \Rightarrow 1)$. We assume that $[\![0,1]\!]_{\alpha,\beta} \cdot [\![1,0]\!]_{\alpha,\beta}$ is an element of M, and we'll show that this one product tells us if M is an origami ring. We know that M is a group under addition by Theorem 2.8. So, $e \in M$ gives:

$$e^2 = e - e + e^2$$

$$e^2 = e - e(1-e).$$

We know that $e - e(1-e) \in M$. Let $e^2 = [\![r,s]\!]_{\alpha,\beta}$. Recall that $M = M_{\mathbb{R}} + M_{\mathbb{R}} \cdot [\![0,1]\!]_{\alpha,\beta}$. To show that $M(U)$ is a subring of $\mathbb{C}$, we use linearity and the intersection formula to show closure under multiplication for some $x, y \in M(U)$, where $x = a + be$ and $y = c + de$, and $a, b, c, d \in M_{\mathbb{R}}$:

$$xy = (a+be)(y+de) = ac + (ad+bc)e + bde^2$$

$$xy = ac[\![1,1]\!]_{\alpha,\beta} + (ad+bc)[\![0,1]\!]_{\alpha,\beta} + bd[\![r,s]\!]_{\alpha,\beta}$$

$$xy = [\![ac + bdr, ac + ad + bc + bds]\!]_{\alpha,\beta}.$$

Due to the ring structure of $M_{\mathbb{R}}$, the $\alpha-$ and $\beta-$ projections of xy are elements of $M - \mathbb{R}$, showing that $xy \in M$, and consequent closure under multiplication for M. Hence we have shown that if $[\![0,1]\!]_{\alpha,\beta} \cdot [\![1,0]\!]_{\alpha,\beta}$ is an element of M, then M is a ring. $\square$

The significance of the first and last statements of Theorem 2.14 cannot be understated. Thus far, we have been understanding our origami sets by step-wise constructing them, and seeing them to determine underlying structure. However, now, we can calculate only first-generation points to determine the *entire* algebraic structure of the origami set. While these statements are powerful on their own, we can use them to elucidate even more criteria for an origami set to be an origami ring.

COROLLARY 2.15. *If M is an origami ring and if $U' \subseteq (0,\pi)$ contains U, then $M(U')$ is an origami ring too. The ring property of M is preserved under extensions of U.*

PROOF. Choose $\alpha, \beta \in U$. Then, since M is an origami ring, by Theorem 2.14, $[\![0,1]\!]_{\alpha,\beta} \cdot [\![1,0]\!]_{\alpha,\beta} \in M$. Since $U \subseteq U'$, it is clear that $M \subseteq M(U')$. Hence $[\![0,1]\!]_{\alpha,\beta} \cdot [\![1,0]\!]_{\alpha,\beta} \in M(U')$, and by Theorem 2.14, $M(U')$ is an origami ring.

$\square$

The next result answers whether every origami set is a subset of an origami ring (yes).

COROLLARY 2.16. *If the set U of prescribed slopes contains $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, then M is an origami ring.*

PROOF. Set $\alpha = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$. Then,

$$\frac{\sin^2 \alpha}{\sin^2(\alpha - \beta)} = 1$$
$$\frac{\sin^2 \beta}{\sin^2(\alpha - \beta)} = 1.$$

We know that $1 \in M_{\mathbb{R}}$. Hence, by Theorem 2.14 $M\left(U \cup \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}\right)$ is an origami ring.

$\square$

Corollary 2.15 and Corollary 2.16 show that by allowing at most two additional slopes, every origami set extends to an origami ring.

Every origami set is contained in an origami ring. Later, we will explore how we might be able to classify these "parent" origami rings.

6. An Example

Recall our previous example of $M(\tilde{U})$, where $\tilde{U} = \left\{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}\right\}$. Earlier, we noted that this is a dense set, and wondered whether a lack of topological structure may imply anything about algebraic structure. We will use the criteria of Theorem 2.14 to show that $M(\tilde{U})$ *is* in fact an origami ring. Set $\alpha := \frac{\pi}{3}$, $\beta := \frac{\pi}{4}$, and $\gamma := \frac{\pi}{5}$. Then,

$$\frac{\sin^2 \alpha}{\sin^2(\alpha - \beta)} = 6 + 3\sqrt{3}$$
$$\frac{\sin^2 \beta}{\sin^2(\alpha - \beta)} = 4 + 2\sqrt{3}.$$

At first glance, this does not seem particularly helpful, but we will use other criteria too. It follows that M is an origami ring if and only if $\sqrt{3} \in M_{\mathbb{R}} = \mathbb{Z}\left[p, \frac{1}{p}, \frac{1}{p-1}\right]$. Here,

$$p := p(\gamma) = \frac{\sin(\alpha - \gamma)\sin\beta}{\sin(\alpha - \beta)\sin\gamma} = \left(1 + \sqrt{3}\right)\sqrt{2 + \frac{2}{\sqrt{5}}}\sin\left(\frac{2\pi}{15}\right).$$

This is the use of $\mathbb{Z}\left[\Delta, \Delta^{-1}\right]$. See that p is algebraic over $\mathbb{Q}$, and the minimal polynomial is given by

$$X^8 + 4X^7 - 8X^6 - 20X^5 + \frac{104}{5}X^4 + 16X^3 - 8X^2 - \frac{16}{5}X + \frac{16}{25} \in \mathbb{Q}\left[X\right].$$

Since $M_{\mathbb{R}} \subseteq \mathbb{Q}(p)$, $\sqrt{3} \in \mathbb{Q}(p)$ is necessary for M to be a ring. Note that $X^2 - 3 \in \mathbb{Q}(p)\left[X\right]$ splits, showing that $\sqrt{3} \in \mathbb{Q}(p)$. We proceed by showing that $\sqrt{3} \in M_{\mathbb{R}}$. Since

$$M_{\mathbb{R}} = \mathbb{Z}\left[p, \frac{1}{p}, \frac{1}{p-1}\right] = \left\{\frac{f(p)}{p^a\left(p-1\right)^b} : f \in \mathbb{Z}\left[X\right] \text{ and } a, b \in \mathbb{N}\right\},$$

we know that $\sqrt{3}$ is an element of $M_{\mathbb{R}}$ if and only if there are $f \in \mathbb{Z}[X]$ and $a, b \in \mathbb{N}$ such that $\sqrt{3}p^a (p-1)^b = f(p)$ is satisfied.

To choose f, a, and b, we choose random parameters a, b and check if $X^2 - 3p^{2a}(p-1)^{2b}$ splits over $\mathbb{Q}(p)$. If so, roots can be represented as polynomials in p with rational coefficients. After a number of steps, we find that $(a, b) = (5, 4)$ yields:

$$\begin{aligned}\sqrt{3} = &\frac{1}{p^5(p-1)^4}(-20p^{13} - 80p^{12} + 140p^{11} + 305p^{10} - 338p^9 + 110p^8 \\ &+ 292p^7 - 194p^6 - 825p^5 + 46p^4 + 242p^3 - 28p^2 - 56p + 8) \in M_{\mathbb{R}}.\end{aligned}$$

By Theorem 2.14, M is an origami ring. Here, we see that M has algebraic structure. But, we have noted that M is a dense set. So, it is possible to have the algebraic structure of a ring without having the topological structure of a lattice. Prior to this example, we have only seen origami sets that are both lattices *and* rings. We now know that an origami lattice is not a prerequisite for an origami ring.

Now that we understand what gives rise to the structure of an origami set in $\mathbb{C}$, we will turn our attention to origami sets in $\mathbb{H}$, and see how the hyperbolic plane might elucidate more about the conditions for structure in an origami set.

CHAPTER 3

The Hyperbolic Plane and Hyperbolic Geometry

1. Introduction to Hyperbolic Geometry

Thus far, we have been working in the complex plane, which is a Euclidean space. Now, Euclidean spaces are made from Euclidean geometry, and Euclidean geometry is familiar to us. Euclidean spaces must follow the postulates of Euclid, which are listed below [**Sta93**] A small explanation is also given below statements to understand their meanings.

(1) To draw a straight line from any point to any point.

    This says that every pair of distinct points can be joined by a straight line. Furthermore, this includes the assumption that two points can be joined by at most one straight line.

(2) To produce (extend) a finite straight line continuously in a straight line.

    This tells us that the plane extends infinitely far in all directions.

(3) To describe a circle with any centre and distance (radius).

Given a point A and a line segment AB, there exists a circle with center A and radius AB.

(4) That all right angles are equal to one another.

The right angle is Euclid's unit for measuring all rectilineal angles, but Euclid did not have a way to prove the congruence of all right angles.

(5) [The parallel postulate]: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles [in sum], the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The parallel postulate can be stated in many different ways. Playfair's expression of the postulate is that given a a Euclidean line L and a point p which is not on L, there exists a unique line through p that is parallel to L. However, in *hyperbolic geometry*, we use the first four axioms, and do not assume the Parallel Postulate. There are two ways in which we can ignore the Parallel Postulate: if there is no line through p that is parallel to L, or more than one line through p that is parallel to L. If there are no lines satisfying that condition, then we find ourselves using elliptical geometry. If there are more than one lines satisfying those conditions, then we find ourselves using hyperbolic geometry. Hence, given a hyperbolic line ℓ and a point p not on ℓ, there are *at least two* hyperbolic lines through p and parallel to ℓ.

One representation of the hyperbolic plane is saddle-shaped, which also has interesting implications for the sum of interior angles in polygons. This introduction to and subsequent discussion of hyperbolic geometry comes from Anderson's book on Hyperbolic Geometry [**And05**].

The hyperbolic plane is simply a plane where the first four postulates hold. Additionally, there are at least two lines through a point p that are parallel to a given line ℓ. This is true in every representation of the hyperbolic plane. There are many representations that we can use for the hyperbolic plane. For our constructions, we will first work with the upper half-plane $\mathbb{H}$ in $\mathbb{C}$, defined as $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$; the upper half-plane looks like the upper half of the complex plane.

In $\mathbb{H}$, a point retains the same notions from $\mathbb{C}$. The angle between two curves in $\mathbb{H}$ is still defined to be the angle between the curves when they are considered to be curves in $\mathbb{C}$, which we know to be the angle between their tangent lines. A Euclidean line $ax + by + c = 0$ in terms of complex coordinates is given by

$$\frac{1}{2}\,(a - ib)\,z + \frac{1}{2}\,(a + ib)\,\overline{z} + c = 0.$$

Though single hyperbolic lines share many properties with single Euclidean lines, we have yet to define a hyperbolic line.

Definition (Hyperbolic Line). A *hyperbolic line* can be defined in two ways, both in terms of Euclidean objects in $\mathbb{C}$:

(1) as the intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$.
(2) as the intersection of $\mathbb{H}$ with a Euclidean circle centered on the real axis $\mathbb{R}$.

Clearly the hyperbolic definition of a line is a little different from a Euclidean line. But, we can see that there are similarities between the two.

THEOREM 3.1. *For each pair p and q of distinct points in $\mathbb{H}$, there exists a unique hyperbolic line ℓ in $\mathbb{H}$ passing through p and q.*

PROOF. There are two cases to consider here. First, suppose that $\mathrm{Re}\,(p) = \mathrm{Re}\,(q)$. Then, the Euclidean line L given by the equation $L = \{z \in \mathbb{C} | \mathrm{Re}\,(z) = \mathrm{Re}\,(q)\}$ is perpendicular to the real axis and passes through both p and q. So, the hyperbolic line $\ell = \mathbb{H} \cap L$ is the hyperbolic line through p and q such that $\mathrm{Re}\,(p) = \mathrm{Re}\,(q)$.

Now, suppose that $\mathrm{Re}\,(p) \neq \mathrm{Re}\,(q)$. Since the Euclidean line through p and q is no longer perpendicular to $\mathbb{R}$, we must construct a Euclidean circle centered on the real axis $\mathbb{R}$ that passes through both p and q.

Let L_{pq} be the Euclidean line segment joining p and q, and let K be the perpendicular bisector of L_{pq}. Then, every Euclidean circle that passes through p and q has its center on K. Since $\mathrm{Re}\,(p) \neq \mathrm{Re}\,(q)$, then the Euclidean line K is not parallel to $\mathbb{R}$. Hence, $K \cap \mathbb{R}$ at a unique point, which we will call c.

Let A be the Euclidean circle centered at c with radius $|c - p|$ to ensure that A passes through p. Since c lies on K, we know that $|c-p| = |c-q|$, so A passes through both p and q. Then, the $\ell = \mathbb{H} \cap A$ is the hyperbolic line through p and q such that $\mathrm{Re}\,(p) \neq \mathrm{Re}\,(q)$. □

When p and q have equal real parts, then we know that $\ell = \mathbb{H} \cap L$, where ℓ is the hyperbolic line in $\mathbb{H}$ passing through p and q, and L is the Euclidean line such that $L = \{z \in \mathbb{C} | \mathrm{Re}\,(z) = \mathrm{Re}\,(q)\}$.

Let p and q be distinct points in $\mathbb{C}$ with non-equal real parts. Define L_{pq} as the Euclidean line segment joining p and q. The midpoint of L_{pq}

is $\frac{1}{2}(p+q)$, and the slope of L_{pq} is $m = \frac{\mathrm{Im}(q)-\mathrm{Im}(p)}{\mathrm{Re}(q)-\mathrm{Re}(p)}$. The perpendicular bisector K of L_{pq} passes through the midpoint, and has slope $-\frac{1}{m}$. So, K has the equation

$$y - \frac{1}{2}\left(\mathrm{Im}(p) + \mathrm{Im}(q)\right) = \frac{\mathrm{Re}\,(q) - \mathrm{Re}\,(p)}{\mathrm{Im}(q) - \mathrm{Im}(p)}\left(x - \frac{1}{2}\left(\mathrm{Re}(p) + \mathrm{Re}(q)\right)\right).$$

The Euclidean center c of A is the $x-$intercept of K, which is:

$$\begin{aligned} c &= \left[-\frac{1}{2}\left(\mathrm{Im}(p) + \mathrm{Im}(q)\right)\right]\left[\frac{\mathrm{Im}(q) - \mathrm{Im}(p)}{\mathrm{Re}\,(q) - \mathrm{Re}\,(p)}\right] + \frac{1}{2}\left(\mathrm{Re}(p) + \mathrm{Re}(q)\right) \\ &= \frac{1}{2}\left[\frac{(\mathrm{Im}(p))^2 - (\mathrm{Im}(q))^2 + (\mathrm{Re}(p))^2 - (\mathrm{Re}(q))^2}{\mathrm{Re}(p) - \mathrm{Re}(q)}\right] \\ &= \frac{1}{2}\left[\frac{|p|^2 - |q|^2}{\mathrm{Re}(p) - \mathrm{Re}(q)}\right] \end{aligned}$$

So, the Euclidean radius of A is $r = |c - p| = \left|\frac{1}{2}\left[\frac{|p|^2-|q|^2}{\mathrm{Re}(p)-\mathrm{Re}(q)}\right] - p\right|$.

Definition (Parallel Hyperbolic Lines). Two hyperbolic lines are *parallel* if they are disjoint.

Now we know what hyperbolic lines are under different circumstances. In the following figure, we have several parallel hyperbolic lines.

2. Origami Constructions in the Hyperbolic Plane: A Direct Transfer?

In terms of constructions, we can think of our intersections using the seed points and the above descriptions of hyperbolic lines. Recall the process for an origami construction: starting with a set of seed points and a prescribed set of angles, and we form new reference points using intersections from extensions along the angles from a pre-existing

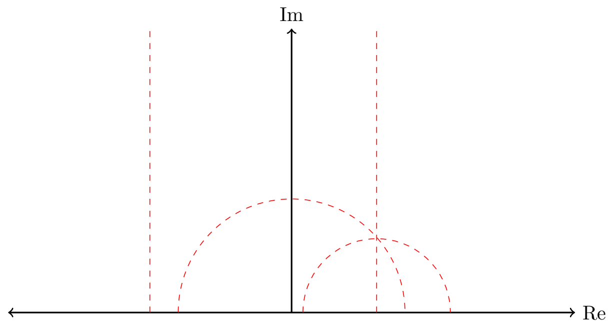


FIGURE 3.1. Several parallel hyperbolic lines: the three hyperbolic lines to the right are all parallel to the hyperbolic line on the left.

reference point.

In the lens of the Euclidean plane, we made the following origami set using the seed points 0 and 1, and the angles $\{0, \frac{\pi}{3}, \frac{2\pi}{3}\}$.

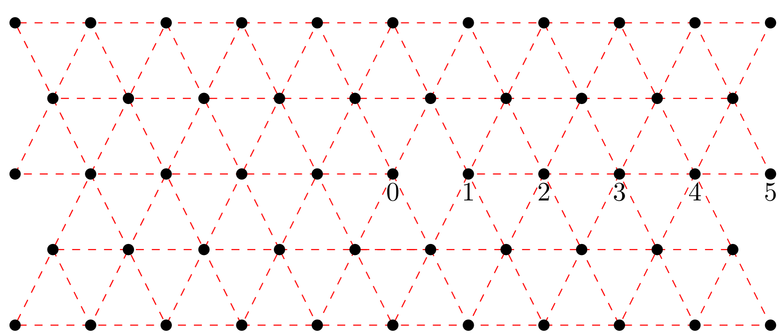


FIGURE 3.2. The origami set constructed from 0 and 1 with angles $\left\{0, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$.

2.1. Euclidean Circles through Seed Points. Now, if we try to use 0 and 1 as points in the hyperbolic plane, we need to make a decision on how those points are involved in creating the correct Euclidean circle to form a hyperbolic line. If the Euclidean circle is chosen to go through a seed point and is centered on the real axis, then we get the following scenario:

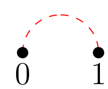


FIGURE 3.3. Attempt #1 at an origami construction in $\mathbb{H}$.

Notice that the angle *through* the seed point is $\frac{\pi}{2}$, which is *not* an allowed angle in this particular construction, and we cannot make any further intersection points. This origami set is just the seed points. We also have not utilized the angles in making this construction, other than to determine that this is not the origami set. So, the hyperbolic line must be something else.

2.2. The Angle through a Seed Point. If we ensure that the angle through a seed point is given by an angle in U, we must determine the center of the Euclidean circle. We must also choose a radius. Then, intersections should be points where these hyperbolic lines intersect. Let's try this on an example.

We will construct $[\![0,1]\!]_{\pi/3,2\pi/3}$, and we choose a radius. Let the radius be 1. So, there is a Euclidean circle that passes through 0, and the tangent line of this curve has a slope of $\frac{\pi}{3}$. So, there is a point of magnitude 1 such that the circle centered at this point passes through

0 with an angle of $\frac{\pi}{3}$. These criteria describe a circle with a center of $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}i\right)$, which we plot in the upper half-plane below. This center was found by analyzing the point on the unit circle where the tangent line to the boundary was $\frac{\pi}{3}$, and translated accordingly so that the boundary was at 0, and the center was shifted to the appropriate point.

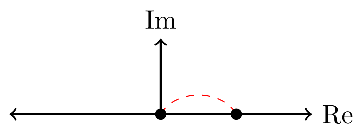


FIGURE 3.4. The Euclidean circle passing through 0 with an angle of $\frac{\pi}{3}$.

This is the Euclidean circle $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\right)^2 = 1^2$.

Then, there is a different Euclidean circle that passes through 1, and the tangent line of this curve has a slope of $\frac{2\pi}{3}$. If, for a moment, we think of this Euclidean circle passing through 0, then this describes the circle with radius 1 and center $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}i\right)$, plotted below in the upper half-plane. The center was found using the same procedure as above.

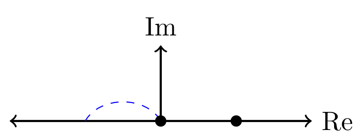


FIGURE 3.5. The Euclidean circle passing through 0 with an angle of $\frac{2\pi}{3}$.

Then, we shift this circle by one unit in the real direction, so we see:

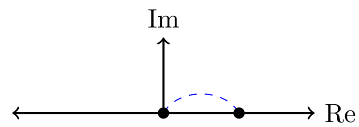


FIGURE 3.6. The Euclidean circle passing through 1 with an angle of $\frac{2\pi}{3}$.

Note that both of these Euclidean circles are the same Euclidean circle, $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\right)^2 = 1^2$, and since they are equal, they have infinitely many intersection points. But, that is far too many intersection points for this to be an origami set.

Clearly, there is much discretion here, and this freedom can be limited by instead understanding a mapping from $\mathbb{C}$ to $\mathbb{H}$ to understand how the origami construction behaves in $\mathbb{H}$. If we know the image of points of a particular construction, and we know how they were formed, then we can begin to understand how the set can be constructed through origami. After that, we can start to understand the algebraic properties of a hyperbolic origami set, and how transformations might change the interactions between points.

3. Mapping from $\mathbb{C} \to \mathbb{D}$

There are many existing maps from $\mathbb{C}$ to $\mathbb{H}$. Instead of the upper-half plane model of the hyperbolic plane, let's use the Poincaré disk, denoted as $\mathbb{D}$. This is another model for two-dimensional hyperbolic geometry on the unit disk. The disk has radius 1, and the boundary is also infinite. The upper half-plane can be thought of as the Poincaré disk with infinite radius. Eventually, the boundary begins to look like a straight line, giving us the upper half-plane. Looking at a previous origami construction, like Figure 3.2, see how this construction is a tiling of $\mathbb{C}$ by triangles. The Poincaré disk can be similarly tiled by triangles, as seen in the following figure.

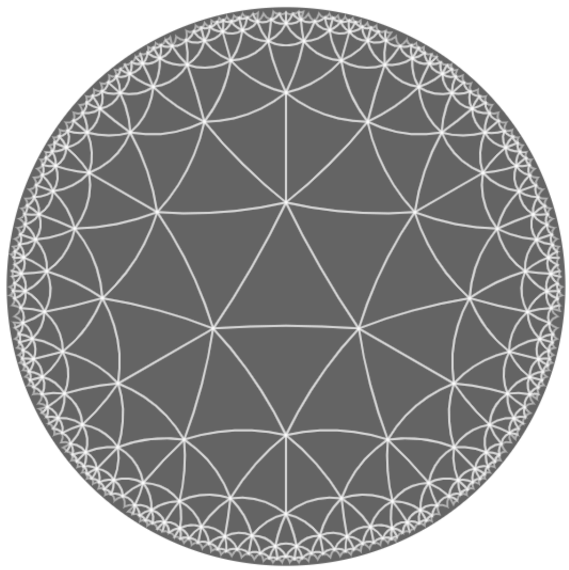


FIGURE 3.7. A tiling of $\mathbb{D}$ by triangles [**Chr20**].

Notice that in Figure 3.7, there are seven equilateral triangles around a vertex. Yet, in Figure 3.2, we see that there are six triangles around a vertex. This is because the sum of interior angles of a hyperbolic triangle is less than 180°, allowing for the extra triangle. In fact, seven is the minimum number of triangle around a vertex. Furthermore, each triangle in Figure 3.7 has the same area [**Sta93**]. Maybe if we see how the folding pattern alters in $\mathbb{D}$, then we can find where the seventh triangle comes from while maintaining the same area.

We'll turn our attention to back the origami set $\mathbb{Z}[i]$, and concentrate on the points generated in the first few generations. We return to $\mathbb{Z}[i]$ because the numbers formed in this set are very simple, making the computations much easier. Using the map from $\mathbb{H}$ to $\mathbb{D}$ given by $\omega = \dfrac{z-i}{z+i}$, we can try to map individual points of an origami construction in $\mathbb{C}$, consider only those with $\mathrm{Im}(z) > 0$. Note that we must still "delete" half of the construction to get to use the map. Some values of the image of the Gaussian integers under this map can be found in the following table, where the original $[\![p,q]\!]_{\alpha,\beta}$ will help us to understand the polygons.

Table 1. Mapping the points of $\mathbb{Z}[i] \cap \mathbb{H}$ to $\mathbb{D}$.

$[\![p,q]\!]_{\alpha,\beta}$	z	ω
$[\![0,0]\!]_{\alpha,\beta}$	0	-1
$[\![1,1]\!]_{\alpha,\beta}$	1	$-i$
$[\![0,1]\!]_{\pi/4,\pi/2}$	$1+i$	$\frac{1}{5}-\frac{2}{5}i$
$[\![1+i,0]\!]_{0,\pi/2}$	i	0
$[\![1+i,1]\!]_{0,\pi/4}$	$2+i$	$\frac{4}{5}-\frac{2}{5}i$
$[\![2+i,1+i]\!]_{\pi/2,\pi/4}$	$2+2i$	$\frac{7}{13}-\frac{4}{13}i$

See that $\mathrm{Im}(1+i) = \mathrm{Im}(2+i)$, and these two points have the same $\mathrm{Im}(\omega)$, possibly giving this map a chance at successfully making a hyperbolic origami set. However, our seed points 0 and 1 turn into a purely real and purely imaginary number, respectively. Looking at the values of the image, the possibility of closure of multiplication looks unlikely, though of course, we have not calculated the entire image of $\mathbb{Z}[i]$. Plotting our ω points in the Poincaré disk, and connecting lines appropriately, we see that we form the following figure.

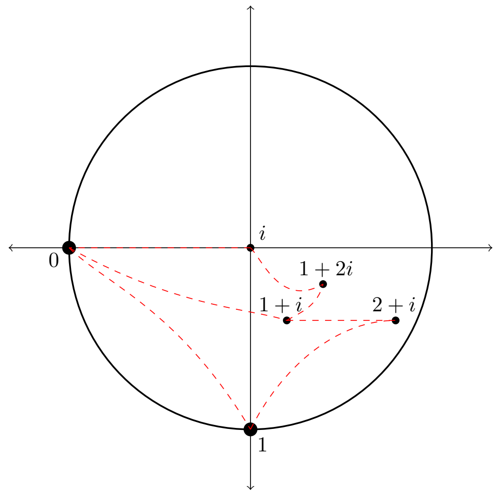


FIGURE 3.8. Image of $\mathbb{H} \to \mathbb{D} : \omega = \frac{z-i}{z+i}$ on $\{z \in \mathbb{Z}[i] \cap \mathbb{H}\}$.

While some liberties were taken in connecting sides of the quadrilaterals of $\mathbb{Z}[i]$, these quadrilaterals are a far cry from our lattice in $\mathbb{C}$. Compare this to the tiling of the Poincaré disk with a minimum number of (equilateral) quadrilaterals, we see:

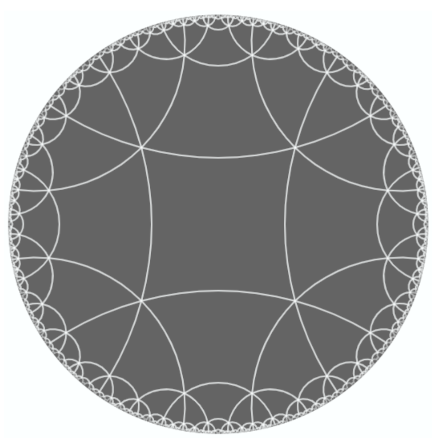


FIGURE 3.9. A tiling of $\mathbb{D}$ by quadrilaterals [**Chr20**].

Furthermore, in this mapping, we must find the image of each constructed point, and reconstruct the polygons using the angles from $\mathbb{C}$. The angles of U, aside from 0, also do not seem make as much of an appearance here.

In our attempts to make a hyperbolic analog of origami constructions in the complex plane, we must choose many more constraints. So, it is clear that there is not a simple correspondence between a complex origami construction and a hyperbolic one. This is likely due to the innate geometry of the hyperbolic plane. We will continue to explore origami sets in the hyperbolic plane, but rather than looking at a direct origami set, we'll classify them using the hyperbolic plane as a modular space.

CHAPTER 4

Origami, Algebra, and the Hyperbolic Plane

1. Revisiting Origami Lattices

Recall that in an origami construction, when we start with three angles where one is 0, by Theorem 2.4, we obtain a lattice. We can represent any general lattice as the integral linear combinations of a set of basis vectors. By Theorem 2.8, we know that an origami set M, can be written as the combination $M_{\mathbb{R}} + M_{\mathbb{R}}[\![0,1]\!]_{\alpha,\beta}$. Since our origami constructions are two-dimensional, an origami lattice is some $\mathbb{Z}z_1 + \mathbb{Z}z_2$. For an origami lattice, we know that $M_{\mathbb{R}} = \mathbb{Z}$ (Theorem 2.4). Hence an origami lattice can be written as $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\alpha,\beta}$.

We will now turn our attention to the behavior of complex lattices in $\mathbb{H}$ to understand the relationship between origami constructions and the hyperbolic plane. While our original goal was to see how the origami constructions might exist in the hyperbolic plane and how hyperbolic geometry might affect the underlying algebra of an origami set, we will instead *classify* origami lattices using the hyperbolic plane. We will use group actions of the classical modular group to guide our classification scheme. Most of the introduction to the classical modular group is from Voight's text *Quaternion algebras* [**Voi18**].

We have worked closely with two origami lattices, one made from $U_1 = \{0, \frac{\pi}{3}, \frac{2\pi}{3}\}$ and one from $U_2 = \{0, \frac{\pi}{4}, \frac{2\pi}{2}\}$. From this point forward, we will consider $[\![0,1]\!]_{\alpha,\beta}$ where $\mathrm{Im}\left([\![0,1]\!]_{\alpha,\beta}\right) > 0$. We also know that both are origami lattices and origami rings. For now, we'll focus on their lattice properties.

2. Lattices

Definition (Homothetic). Two lattices are *homothetic* if there exists $u \in \mathbb{C} \setminus \{0\}$ such that $\Lambda' = u\Lambda$. Then, we write $\Lambda \sim \Lambda'$.

Lemma 4.1. *Homothety of lattices is an equivalence relation.*

Proof. We will show that all three parts of the definition of an equivalence relation are satisfied.

(1) First, we know that $\Lambda \sim \Lambda$ because $\Lambda = 1\Lambda$, and $1 \in \mathbb{C} \setminus \{0\}$. So homothety is reflexive.
(2) Next, if $\Lambda \sim \Lambda'$, then there exists u such that $\Lambda = u\Lambda'$. It follows that $\frac{1}{u}\Lambda = \Lambda'$ because $u \in \mathbb{C} \setminus \{0\}$, so $\frac{1}{u}$ is well-defined, and $\Lambda' \sim \Lambda$.
(3) Finally, if $\Lambda \sim \Lambda'$, and $\Lambda' \sim \Lambda''$, then we know that $\Lambda = u'\Lambda'$ and $\Lambda' = u''\Lambda''$. So, $\Lambda = u'(u''\Lambda'') = u'u''\Lambda''$. We know that $u'u'' \in \mathbb{C} \setminus \{0\}$, so $\Lambda \sim \Lambda''$.

$\square$

Since $\Lambda = \mathbb{Z}z_1 + \mathbb{Z}z_2 = \mathbb{Z}z_2 + \mathbb{Z}z_1$, assume that $\mathrm{Im}\left(\frac{z_2}{z_1}\right) > 0$. Then there is a homothety $\Lambda = \mathbb{Z} + \mathbb{Z}\tau \sim \mathbb{Z} + \mathbb{Z}\tau' = \Lambda'$. Here, $\tau = \frac{z_2}{z_1} \in \mathbb{H}$. Since $\mathrm{Im}\left(\frac{z_2}{z_1}\right) > 0$, then $\{z_1, z_2\}$ is an oriented basis, i.e. the basis is asymmetric in a way that makes it impossible to recreate a reflection through rotations only. A homothety is a transformation of space that dilates distances. For our origami sets, we can see that $\tau = [\![0,1]\!]_{\alpha,\beta}$.

Before we see conditions that tell us when lattices are homothetic, we'll look at the classical modular group.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is the special linear group of 2×2 matrices with integer entries and determinant 1. Additionally, $\mathrm{SZ}_2(\mathbb{Z})$ is the subgroup of scalar transformations with determinant 1. The projective special linear group $\mathrm{PSL}_2(\mathbb{Z})$ is the quotient group $\mathrm{SL}_2(\mathbb{Z})/\mathrm{SZ}_2(\mathbb{Z})$. We name $\mathrm{PSL}_2(\mathbb{Z})$ as the classical modular group, and we can also define it as follows:

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} / \{\pm 1\}.$$

See that nothing acts trivially on an element of $\mathbb{H}$ except for the identity matrix.

Define

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

For some $z \in \mathbb{H}$, we see that S acts to "invert" a point into the unit circle, and T acts to translate z one unit to the right. Also note

$$SS = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \in \mathrm{PSL}_2(\mathbb{Z}).$$

Additionally,

$$ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

and

$$(ST)^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1.$$

We'll put these thoughts aside for a moment, and try to understand the action of $\mathrm{PSL}_2(\mathbb{Z})$.

Consider the fundamental domain, shown below in Figure 4.1. We'll denote the fundamental domain as $\sqcup$. The fundamental domain is given by

$$\sqcup = \{z \in \mathbb{H} : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \text{ and } |z| \geq 1\}.$$

See that $\sqcup$ is a hyperbolic triangle with vertices at $\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\omega^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and ∞.

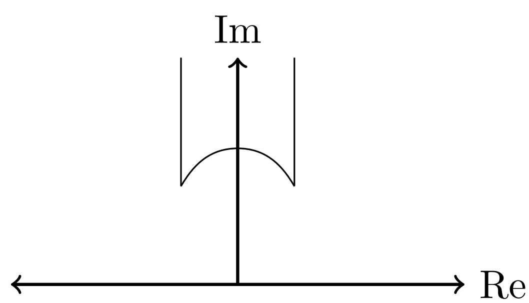


FIGURE 4.1. The fundamental domain, $\sqcup$, in $\mathbb{H}$.

Applying S and T to the vertices tesselate $\sqcup$. In the following figure, we see that $\sqcup$ is denoted by 1, and words in S, T denote the action.

We'll now explore the relationship of $\sqcup$ to Γ.

LEMMA 4.2. *For all $z \in \mathbb{H}$, there exists a word $\gamma \in \langle S, T \rangle$ such that $\gamma z \in \sqcup$.*

PROOF. We will determine this word using a reduction algorithm. Translate z so that $|\mathrm{Re}(z)| \leq \frac{1}{2}$. If $|z| \geq 1$, then we are done. If $|z| \leq 1$,

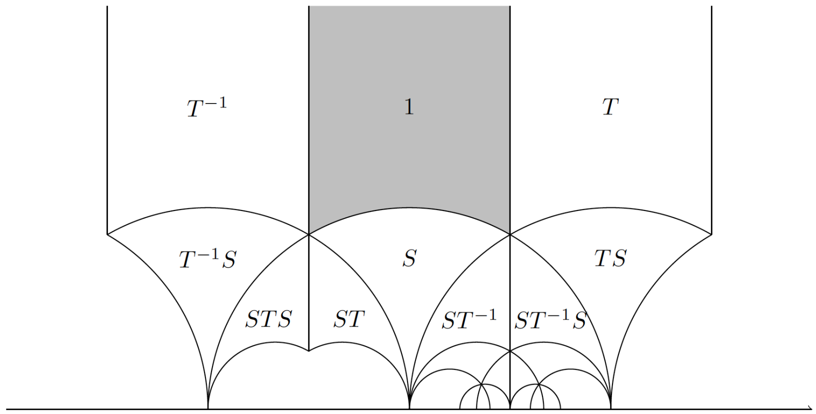


FIGURE 4.2. Tesselations of $\sqcup$ by words S, T [**Voi18**].

then

$$\operatorname{Im}\left(-\frac{1}{z}\right) = \frac{\operatorname{Im}(z)}{|z|^2} > \operatorname{Im}(z).$$

By repeating this process, we develop a sequence $z_1, z_2, \ldots, z_n$, and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2) < \cdots < \operatorname{Im}(z_n)$. This is completed in a finite number of steps because we know that

$$\operatorname{Im}(gz) = \frac{\operatorname{Im}(z)}{|cz+d|^2}, g \in \mathrm{PSL}_2(\mathbb{Z}),$$

and the number of $c, d \in \mathbb{Z}$ such that $|cz + d| < 1$ is finite. There are only finite elements of this bound because $\mathbb{Z} + \mathbb{Z}z$ is a complex lattice. When this algorithm terminates, we have a word γ in $\langle S, T\rangle$ such that $\gamma z \in \sqcup$, completing our proof.

$\square$

So, we now know that we can make our basis point z of our lattice in $\mathbb{H}$ go into the fundamental domain.

LEMMA 4.3. *Let $z, z' \in \sqcup$, and suppose $z \in int(\sqcup)$ lies in the interior of $\sqcup$. If $z' = \gamma z$ with $\gamma \in \Gamma$, then $\gamma = 1$ and $z = z'$.*

PROOF. From our assumptions, we see that

$$\mathrm{Im}(z') = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

This is a proof by contradiction.

First, let's suppose that $\mathrm{Im}(z') \geq \mathrm{Im}(z)$. Then,

$$|cz+d|^2 = (c\mathrm{Re}(z)+d)^2 + c^2\,(\mathrm{Im}(z))^2 \leq 1.$$

Since $\mathrm{Im}(z) > \mathrm{Im}(\omega) = \frac{\sqrt{3}}{2}$, we know that $c^2 \leq \frac{4}{3}$, and clearly $|c| \leq 1$. Now, if $c = 0$, then $ad - bc = ad = 1$, and $a = d = \pm 1$. Hence $z' = \gamma z = z \pm b$, which implies $b = 0$ and $\gamma = 1$. Alternatively, if $|c| = 1$, then

$$\begin{aligned} (c\mathrm{Re}(z)+d)^2 + c^2\,(\mathrm{Im}(z))^2 &\leq 1 \\ (c\mathrm{Re}(z)+d)^2 &\leq 1 - (\mathrm{Im}(z))^2 \\ (c\mathrm{Re}(z)+d)^2 &\leq 1 - \frac{3}{4} \\ (c\mathrm{Re}(z)+d)^2 &\leq \frac{1}{4}. \end{aligned}$$

We also know that $|\mathrm{Re}(z)| < \frac{1}{2}$. These two previous inequalities imply that $d = 0$, so $|cz+d| = |z| \leq 1$, and we know $z \in \mathrm{int}(\sqcup)$, so $|z| > 1$. This is our first contradiction.

Now, let's assume that $\mathrm{Im}(z') < \mathrm{Im}(z)$. By the same argument, we reach $|\mathrm{Re}(z)| \leq \frac{1}{2}$, and consequently, $|z| < 1$, our second contradiction. So if $z' = \gamma z$ with $\gamma \in \Gamma$, then $\gamma = 1$ and $z = z'$.

$\square$

We are now ready to show that S and T alone generate $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

LEMMA 4.4. *The elements S and T generate $\Gamma = PSL_2(\mathbb{Z})$.*

PROOF. Let $z = 2i \in \mathrm{int}(\sqcup)$. Let $\gamma \in \Gamma$, and let $z' = \gamma z$. By Lemma 4.2, there exists a word $\gamma' \in \langle S, T \rangle$ such that $\gamma' z' \in \sqcup$. By Lemma 4.3, we know that $\gamma' z' = (\gamma'\gamma) z = z$, so $\gamma'\gamma = 1$ and $\gamma = \gamma' \in \langle S, T \rangle$.

□

Hence $\mathrm{PSL}_2(\mathbb{Z})$ has the presentation $\langle S, T | S^2 = (ST)^3 = 1 \rangle$.

COROLLARY 4.5. *The set $\sqcup$ is a fundamental set for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$.*

We'll use our previous lemmas and discussion of Γ to understand that $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is a moduli space of complex lattices.

LEMMA 4.6. *Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and $\Lambda' = \mathbb{Z} + \mathbb{Z}\tau'$ be lattices with $\tau, \tau' \in \mathbb{H}$. Then $\Lambda \sim \Lambda'$ if and only if there exists some $A \in SL_2(\mathbb{Z})$ such that $\tau' = A\tau$.*

PROOF. First, let's prove the forward direction. Let's assume that Λ and Λ' are homothetic. Then there exists some $u \in \mathbb{C} \setminus \{0\}$ such that $\Lambda = u\Lambda'$. So, u and $u\tau'$ generate Λ, so there exists $g \in \mathrm{PSL}_2(\mathbb{Z})$, where $\alpha\tau' = a\tau + b$, and $\alpha = c\tau + d$. Then,

$$\tau \ = \frac{a\tau + b}{\alpha} = \frac{a\tau + b}{c\tau + d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tau.$$

Since τ and τ' are both in $\mathbb{H}$, then $ad - bc = 1$, so $g \in \mathrm{SL}_2(\mathbb{Z})$. Conversely, suppose there is some $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, such that $\tau' = A\tau$. Let $\alpha = c\tau + d$. Then, $\alpha\tau' = a\tau + d$, so $\Lambda' \subseteq \alpha\Lambda$. Since $ad - bc = 1$, then $\tau = A^{-1}\tau'$, and $A^{-1} \in \mathrm{SL}_2(\mathbb{Z})$, showing that $\alpha\Lambda \subseteq \Lambda'$.

So, Λ is homothetic to Λ'.

$\square$

Lemma 4.6 tells us that there is a bijection

$$Y = \Gamma\backslash\mathbb{H} \to \{\Lambda \subset \mathbb{C}\}/\sim \Gamma\tau \mapsto [\mathbb{Z} + \mathbb{Z}\tau].$$

This is telling us that $Y = \Gamma\backslash\mathbb{H}$ parameterizes a class of equivalent complex lattices up to homothety. In this map, Y, the quotient $\Gamma\backslash\mathbb{H}$ corresponds to a complex lattice, which is related by homothety under $\Gamma\tau$. Such homothetic lattices are found through $\Gamma\tau$. Since $\sqcup$ is the fundamental domain for the group action of Γ on $\mathbb{H}$, we need $\gamma\tau$ to be within $\sqcup$.

As it turns out, to explore the behavior of origami sets in the hyperbolic plane, we do not go through the origami procedure. Instead, we think about the bijection given above that tells us where homothetic lattices are sent in the fundamental domain. We'll explore some homothetic origami lattices, and see their image in $\sqcup$, and raise some new questions and ways to think about answering them.

3. Finding Homothetic Lattices

Thinking about the origami lattices that we have worked with most, $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/3,2\pi/3}$ and $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/4,\pi/2}$, we could try to see if they are homothetic and where they might be sent under $\Gamma\backslash\mathbb{H}$. In short, these two origami lattices are not homothetic, because we know that $[\![0,1]\!]_{\pi/3,2\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, while $[\![0,1]\!]_{\pi/4,\pi/2} = 1 + i$. There are no $\gamma, \gamma' \in \Gamma$ such that $\gamma \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \gamma' \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because we will never be able to generate the $\sqrt{3}$ if the entries of γ are integers. So, we'll have to creatively explore other angles that might lead to homothetic origami lattices.

Being creative in this way is very difficult, because we are trying to find homotheties. But, we must keep in mind that the entries of γ must be integers, and γ must have a determinant of 1. For example, trying the two origami lattices given by $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/4,\pi/2}$, and $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/4,3\pi/4}$, we find ourselves trying to solve the following equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Since $\Gamma\tau = \Gamma\tau'$, we should be able to find the right γ' for any chosen γ. So, let $\gamma = S$. Then

$$S[\![0,1]\!]_{\pi/4,\pi/2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Unfortunately, for us, then we have

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a'}{2} + \frac{b'}{2} \\ \frac{c'}{2} + \frac{d'}{2} \end{bmatrix}.$$

There are no values of $a', b' \in \mathbb{Z}$ that will satisfy both the condition of the determinant and the condition provided by $\gamma\tau$. So, these two origami lattices are not homothetic.

4. Classification Examples

Despite not having a homothety, we can still use Y. We can map the lattice $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/4,\pi/2}$ to the point i in the fundamental domain, using $\gamma = ST$. Indeed,

$$ST[\![0,1]\!]_{\pi/4,\pi/2} = ST \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which is the point i, which is in the fundamental domain $\sqcup$. Similarly, $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/4,3\pi/4}$ can be sent to $-\frac{1}{2} + i$ because

$$ST \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix},$$

which is in the fundamental domain, so $\gamma = 1$. If we think about another of our previous examples, $\mathbb{Z} + \mathbb{Z}[\![0,1]\!]_{\pi/3,2\pi/3}$, then $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}$, which is already in the fundamental domain. None of these lattices are sent to the same point in the fundamental domain, and none of these lattices are homothetic.

Additionally, if we take a random element of $\mathrm{PSL}_2(\mathbb{Z})$ and apply it to $\tau = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, of the Gaussian integers, then we should be able to find some τ' that is homothetic. Consider

$$\gamma = \begin{bmatrix} -3 & -1 \\ -5 & -2 \end{bmatrix},$$

a randomly generated element of Γ produced by Sage [**The20**]. Then,

$$\gamma\tau = \begin{bmatrix} -3 & -1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}.$$

So, we know that $\Lambda_1 = Z + \mathbb{Z}(-4-7i)$ is homothetic to $\Lambda_2 = \mathbb{Z}+\mathbb{Z}[1+i]$. Is Λ_1 an origami lattice? We'll return to this example.

Looking at some previous work, there are two ways to determine the form of an origami set depending on the angles. There are special lattices that are rings of integers of imaginary quadratic fields. First, let's remind ourselves of what algebraic integers are.

Definition (Algebraic Integer [**Mar77**]). A complex number is an *algebraic integer* if and only if it is a root of some monic polynomial with coefficients in $\mathbb{Z}$.

Another type of classification of origami sets is by their ring structure. We can determine if particular subrings of $\mathbb{C}$ are origami rings. We have a result about algebraic integers in quadratic fields.

Theorem 4.7 ([**Mar77**]). *Let m be a squarefree integer. The set of algebraic integers in the quadratic field $\mathbb{Q}[\sqrt{m}]$ is:*

(1) $\{a + b\sqrt{m} : a, b \in \mathbb{Z}\}$ *if* $m \equiv 2$ *or* $3 \mod 4$, *or*
(2) $\left\{\frac{a+b\sqrt{m}}{2} : a, b \in \mathbb{Z},\ a \equiv b \mod 2\right\}$ *if* $m \equiv 1 \mod 4$.

The main result of Kritschgau and Salerno's work, that it is possible to obtain the ring of integers of an imaginary quadratic field through an origami construction, then follows.

Theorem 4.8 ([**KS17**]). *Let $m < 0$ be a squarefree integer, and let $\theta = Arg\,(1 + \sqrt{m})$. Then $\mathcal{O}\,(\mathbb{Q}\sqrt{m}) = M(U)$, where*

(1) $U = \left\{0, \dfrac{\pi}{2}, \theta\right\}$ *if* $m \equiv 2$ *or* $3 \mod 4$, *or*
(2) $U = \{0, \theta, \pi - \theta\}$ *if* $m \equiv 1 \mod 4$.

Note that the two origami sets given by each part of the theorem are not homothetic, because we cannot find something in $\mathrm{SL}_2(\mathbb{Z})$ to transform one into the other, because the origami sets produced by Theorem 4.8.1 correspond to the imaginary quadratic fields of the form of Theorem 4.7.1 while origami sets produced by Theorem 4.8.2 correspond to the imaginary quadratic fields of the form Theorem 4.7.2. No integral and unit determinant matrix can transform $a + b\sqrt{m}$ into $\frac{a+b\sqrt{m}}{2}$. Naturally, we wonder if origami lattices satisfying the different conditions of Theorem 4.8.1 might be homothetic.

Let's use two examples of origami lattices of the form given in Theorem 4.8. Notice that these are indeed lattices by Theorem 2.4. If we consider $m = -3$, then $\theta = \operatorname{Arg}\left(1 + \sqrt{3}i\right) = \frac{\pi}{3}$. So, our set of angles is $U = \{0, \frac{\pi}{2}, \frac{\pi}{3}\}$. With seed points 0 and 1, then $[\![0,1]\!]_{\pi/2,\pi/3} = 1 + \sqrt{3}i$. So, $\Lambda = \mathbb{Z} + \mathbb{Z}(1 + \sqrt{3}i)$, and hence $\tau = 1 + \sqrt{3}i$. We can map τ to a point in $\sqcup$. See that

$$TTS \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{3} \\ 1 \end{bmatrix}.$$

Now, consider $m = -2$, then $\theta = \operatorname{Arg}\left(1 + \sqrt{2}i\right) = \arctan\left(\sqrt{2}\right)$. Then, we have that $[\![0,1]\!]_{\pi/2,\arctan\left(\sqrt{2}\right)} = 1 + \sqrt{2}i$. See that

$$TTS \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{2} \\ 1 \end{bmatrix}.$$

So, despite these two origami lattices having the same form by Theorem 4.8, they are not homothetic.

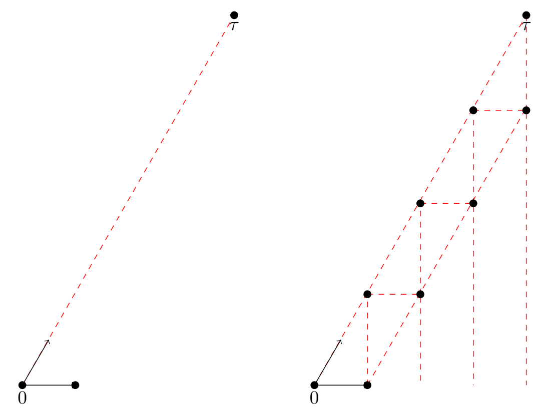
5. Conjectures and Further Questions

These examples raise questions. First, if an origami lattice Λ is also an origami ring, and Λ is homothetic to another origami lattice Λ', is Λ' also an origami ring? We have found different expressions for the same lattice, which should be related by homothety. Since they produce the same origami set, the ring structure should be preserved in that case.

Lastly, this work begs the question whether all lattices are origami lattices. Clearly, we know that all origami lattices are indeed lattices. Recall our example of $\mathbb{Z} + \mathbb{Z}[1+i] = \Lambda_2 \sim \Lambda_1\mathbb{Z} + \mathbb{Z}[4+7i]$. Clearly, Λ_1 is not the Gaussian integers, but they are homothetic. We wondered if Λ_1 is an origami lattice. See that we cannot perform our origami process without a basis of length 1. Let our basis for this lattice Λ_1 be given by $\{1, \frac{4}{\sqrt{65}} + \frac{7}{\sqrt{65}}\}$. Then, we get the scenario in figure below.

FIGURE 4.3. The basis of $\Lambda_1 = \mathbb{Z} + \mathbb{Z}[4+7i]$, with τ included.

Let $U = \{0, \frac{\pi}{2}, \arg\tau.\}$. Then, we may construct, through a sequence of monomials, the point τ using the angles of U.



(a) How τ relates to the basis.

(b) The monomial sequence.

FIGURE 4.4. The sequence of monomials to reach τ.

Along the way of this monomial sequence, we make a number of points of $M(U)$ that are not elements of Λ_1. We see that we can make $\Lambda_1 \subset M(U)$, but it is not possible to eliminate the extra points. We hypothesize that any lattice $\Lambda = \mathbb{Z}+\omega\mathbb{Z}$, we can find $\Lambda \subset M(U)$ where $U = \{0, \frac{\pi}{2}, \arg z\}$. This may relate to the idea of maximal orders. We further believe that if a lattice, Λ, is not maximal, then $\Lambda \subset M(U)$. However, if a lattice Λ is maximal, then $\Lambda \subseteq M(U)$. This may mean that every lattice is contained in an origami lattice, and hence origami lattices are maximal orders.

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