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Fluid Stirring on a Sphere -- a Topological Approach

Kirstin E. Koepnick
Bates College, kkoepnic@bates.edu

Fluid Stirring on a Sphere
A Topological Approach

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Kirstin E. Koepnick
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Introduction

From milk in your morning coffee to the air around us, fluid mixing happens everywhere in physical life yet is one of the most difficult mathematical and physical concepts to understand and solve. The basic equations for fluid mechanics are framed in terms of velocity fields and are typically very complicated partial differential equations that are remarkably difficult to solve. Since the formulas for fluid motions are essentially nonexistent, we can employ mathematical tools and algebraic structures in order to understand the mixing of fluids.

Mathematical structures appear in most disciplines and take form in various facets. We can think of structure as something we apply to a set; structure represents the additional information, restrictions, and properties of a set. Furthermore, we can relate sets and structures (groups) through different types of morphisms. To have a stronger relation between groups, we can consider isomorphisms which is the bijective version of morphisms. From a topological standpoint, to relate topological spaces, we look at homeomorphisms and diffeomorphisms as well as homotopy.

We can create a fluid map that describes the evolution of a fluid as the permutation of particles up to a fixed, arbitrary time, T. The complete evolution of this fluid is the entire family of maps, one for each time [**5**]. We can describe two-dimensional flow complexity using basic differential equations

$$\frac{dx}{dt} = u(x, y, t), \frac{dy}{dt} = v(x, y, t) \tag{0.1}$$

where u and v satisfy sets of equations describing fluid motion. So, first introduced by Boyland et al., we can introduce a sort of mathematical structure in order to understand basic kinematics of fluid behavior [**6**].

Although it is interesting to investigate the overall flow of a fluid, in this thesis, we will investigate stirring by employing mathematical structure: braid and mapping class groups. We consider the region occupied by arbitrary viscous fluid as the two-dimensional disk, D^2. By introducing a "mixing device" consisting of independent, permutatble rods as punctures on the disk, D^2, (where each puncture is representative of a mixing or stirring rod) we can stir and therefore mix our fluid. We suppose this mixing device to be a mechanical stirrer that after you stir, the rod stops at the same point as it started or the positions remain the same (i.e. the first may end in the third rod's position but the three have the same starting and ending places). These rods act as topological obstacles that stretch the fluid elements.

Since mixing is extremely difficult to understand and study, many have used braids and braid groups in order to simplify fluid problems. The example of mixing milk in tea or coffee is a rather poor example to use since mixing is achieved almost instantaneously with just a simply flick of the wrist. Naturally, we can make this example more complex. So first, let's consider an arbitrary mixing device that consists of movable mixing rods in a fluid which is contained within a disk boundary.

The stirring and mixing of a fluid with moving rods is vital in many physical applications in order to achieve homogeneity within the mixture. These rods act as an obstacle whose motion stretches and folds together fluid elements [**18**]. Over time, the permutation of these rods comprise a mathematical braid whose properties dictate the *topological entropy*, a number to describe the total disorder or chaos of a system. A braid with topological entropy greater than one exhibits chaotic behavior which guarantees a good mixing of the fluid [**18**]. These rod stirring braids have been previously studied on both the disk as well as the two dimensional torus. Using integral laminations and the counting of intersections, we can estimate the topological entropy of braids on the torus and can therefore enforce chaos by choosing the braid with the highest topological entropy [**15, 9**]. However, the trajectory of fluid mixing on a sphere poses an intriguing starting inquiry to overall mixing on spherical surfaces like the ocean, stars, etc. We use a recipe established by Yvon Verbern to create pseudo-Ansov maps on a punctured sphere using Dehn twist in order to construct similar maps on a 4-times punctured sphere, $S_{0,4}$ [**19**]. Since a quotient of T^2 under the hyperelliptic involution is the 2 sphere with 4 marked points, we are able to use various methods to estimate the topological entropy of a stirring protocol on a 4-times punctured sphere.

In this thesis, we will outline the necessary mathematical concepts needed including braid groups and mapping class groups. From here, we will give an introduction to how fluid stirring can be expressed as a mathematical braid as well as how mapping class groups and topological structures have previously been utilized to estimate topological entropy. We will then introduce Verberne's recipe and apply it to stirring patterns on the sphere [**19**].

CHAPTER 1

Mathematical Preliminaries

1. Group Theory

This section is largely drawn from *A Book of Abstract Algebra* by Charles C. Pinter [**17**]. For this thesis, we assume that the reader is familiar basic group theory and the notion of "open" sets.

The basic principles behind group theory is the concept of adding mathematical structure to sets and understanding how these structures can be related. One of the fundamental structures that can be applied to a set is the idea of a *group*.

Definition. Consider a set G. This set, G, together with a binary operation $\star$ becomes a group if it satisfies these three axioms:

(1) $\star$ is associative
(2) There is an element e such that for every element $g \in G$, we have $g \star e = g$ and $e \star g = g$.
(3) For every element $g \in G$, there exists an element $g^{-1} \in G$ such that $g \star g^{-1} = g^{-1} \star g = e$.

Recall that we represent a group as $\langle G, \star \rangle$ which simply describes the set and the operation applied such that G becomes a group. Possible operations include addition, multiplication, division, etc.

1.1. Subgroups. A *normal* subgroup, denoted $H \triangleleft G$, is a group for which $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$ given group G and subgroup H. An important normal subgroup is the center of a group, denoted $Z(G)$. The center of a group, G, is the group of all elements that commute. In other words, the center is described as

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}. \tag{1.1}$$

Example 1. $\langle \mathbb{Q}^*, \cdot \rangle$ is a subgroup of $\langle \mathbb{R}, + \rangle$ where $\mathbb{Q}^*$ is the group of non zero rational numbers under multiplication.

1.2. Functions and morphisms. Recall that a function, $f : D \to C$, is *injective* (or one-to-one) if every element of C is the image of no more than one element of D. In addition, this function f is *surjective* (or onto) if each element of C is the image of at least one element of D (the function covers the co-domain). Then, a function is *bijective* if it is both injective and surjective.

Definition. Let $\langle A, \star_1 \rangle$ and $\langle B, \star_2 \rangle$ be groups and $f : A \to B$ be a bijective function such that

$$f(a \star_1 b) = f(a) \star_2 f(b)$$

for elements $a, b \in A$ and $f(a), f(b) \in B$ is called an **isomorphism** from A to B denoted $A \simeq B$. A **homomorphism** is an onto function rather than a bijective function with the same properties as an isomorphism.

2. The Fundamental Homomorphism Theorem

Consider groups G and H. We know that H is a homomorphic image of G if and only if H is a quotient group of G. Recall:

Definition. Let $f : G \to H$ be a homomorphism with kernel K. Then

$$f(a) = f(b) \text{ if and only if } Ka = Kb.$$

Then it follows:

Theorem 1.1. ***The Fundamental Homomorphism Theorem****.* *Let $f : G \to H$ be a homomorphism of G onto H. If K is the kernel of f, then*

$$H \cong G/K.$$

3. Exactness

As a consequence of the fundamental homomorphism theorem, we can understand the notion of exactness as well as introduce the short exact sequence of groups.

Suppose G_1 and G_2 are groups with homorphism $\varphi : G_1 \to G_2$. Suppose that φ is surjective. Then there is also a map from the kernal of φ to G_1 through an inclusion map. In other words, we have that

$$1 \xrightarrow{\theta_1} Ker(\varphi) \xrightarrow{i} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\theta} 1$$

For *exactness* to occur at G_1, our outputs from i must be the inputs for φ. In otherwords, $i(Ker(\varphi)) = Ker(\varphi)$. Furthermore, for exactness at G_2, $\varphi(G_1) = Ker(\theta)$ and similarly for exactness at $Ker(\varphi)$, $\theta_1(1) = 1 = Ker(i)$.

In general, we call a short exact sequence of groups A, B, C to be defined as

$$1 \xrightarrow{\theta_1} A \xrightarrow{i} B \xrightarrow{\varphi} C \xrightarrow{\theta} 1$$

such that $B/A \simeq C$ and i is injective and φ is surjective.

4. Topology and Continuity

This section is largely drawn from *Essential Topology* by Martin D. Crossley [**8**]. A topology is merely a structure, like a group, placed on a set that is a collection of distinguished subsets called open sets. The definition for an open set is rather ambiguous when not put within context of a specific topological space; however, in most context, an open set is defined as an allowable neighborhood of a point.

DEFINITION. A *topological space* is a set, X, together with a collection, $\mathcal{T}$, of subsets of X called "open" sets, which satisfy the following rules:

1. The set X itself is "open".
2. The empty set is "open".
3. Arbitrary unions of "open" sets are "open".
4. Finite intersections of "open" sets are "open".

Analogous to continuous functions in calculus or multivariable calculus, we can create continuous functions between two topological spaces. A function $g : X \to Y$ where X, Y are topological spaces is *continuous* if the preimage of every open set in Y is open in X. Formally:

DEFINITION. A function $g : X \to Y$ from one topological space to another is *continuous* if the preimage, $f^{-1}(Q)$ of every open set $Q \subset Y$ is also an open set in X.

In the rest of this thesis, the word "map" and "continuous function" will be used interchangeably.

5. Homeomorphisms

We can relate topological spaces using the notion of homeomorphisms. Concretely, we have:

DEFINITION. Two topological spaces S and T are *homeomorphic* if there are continuous maps $f : S \to T$ and $f^{-1} : T \to S$ such that

$$(f \circ f^{-1}) = id_T \quad \text{and} \quad (f^{-1} \circ f) = id_s.$$

The individual maps f (and f^{-1}) are *homeomorphisms* and the topological spaces are writen then as $S \cong T$.

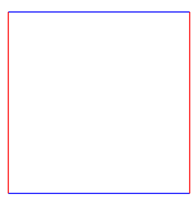
6. Quotient Spaces

One important type of topological space is the *quotient space*.

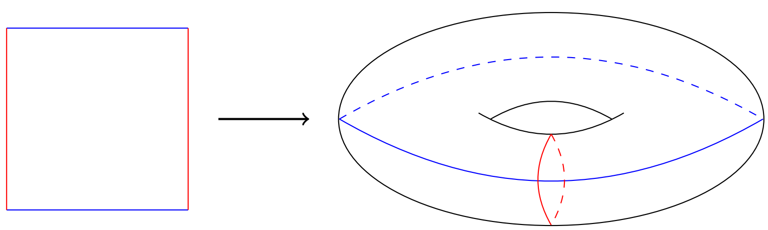
DEFINITION. If X is a topological space with $A \subset X$, then the *quotient space* denote X/A is the set $(X - A) \coprod \{*\}$ where $\{*\}$ is a distinguished point.

Subsets of the quotient space are open only if they are open sets in $(X - A)$ or unions of $\{*\}$ and the intersection with $X - A$ of an open set in X containing A [**8**].

EXAMPLE 2. Consider the square defined by $S = [0,1] \times [0,1]$. Identify opposite sides as



We can see that through this equivalence relation, $S/$, we have



Therefore, the torus is a quotient space. We can also identify the 2 holed torus in a similar way but with an octagon (sides identified) rather than a square.

7. Paths, Loops and Homotopy

7.1. Paths and Loops. A *path* is like a parametric curve as learning in multivariable calculus or linear algebra. An example of a path is: consider the function given by $\gamma : [0,1] \to \mathbb{R}^2$ given by $\gamma(t) = (t, 0)$. This produces a straight line from $(0,0)$ to $(1,0)$ as shown in Figure 1.1.

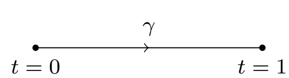


FIGURE 1.1. The path of $\gamma : [0,1] \to \mathbb{R}^2$ given by $\gamma(t) = (t, 0)$

A *loop* is a path that connects at a "base point". An example of a loop is: consider $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. This function forms the unit circle as shown in Figure 1.2.

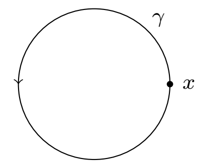


FIGURE 1.2. This is an example of a loop described by the equation $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$ with base point x.

If every loop in a topological space X can be continuously shrunk to a single point, then X is simply connected.

Now, consider two paths f and g. The "addition" of paths is similar to function composition in that we first follow the path of f and then g or vise versa. Formally, the *composition* of two paths, denoted $f \circ g$ is defined as:

$$f \circ g(t) = \begin{cases} g(2t) & 0 \leq t \leq 1/2 \\ f(2t-1) & 1/2 < t \leq 1 \end{cases}.$$

We can see that in order to fit the definition of a path, we must "trace" g and f twice as fast as before we composed the two functions.

7.2. Homotopy. Two maps of topological spaces are considered "similar" if one can be continuously deformed into the other. In other words, consider two fixed points x and y and paths f and g both with endpoints of x and y all within some topological space X. We say that there exists a "homotopy" of paths in X if there exists a family of paths that can be described by $f_t : [0,1] \to X$ for $0 \leq t \leq 1$ such that $f_t(0) = x$ and $f_t(1) = y$. In a sense, this family of paths describes a continuous deformation from some base path f to end path g keeping endpoints fixed.

Formally, we say that spaces are "homotopy equivalent" or there exists a "homotopy" between two topological spaces.

DEFINITION. Two maps $f, g : S \to T$ are *homotopic* if there is a continuous function

$$F : S \times [0,1] \to T$$

such that $F(s,0) = f(s)$ for all $s \in S$ and $F(s,1) = g(s)$ for all $s \in S$. In this case, F is a *homotopy* between f and g and we write $f \simeq g$.

Like most things in abstract algebra and group theory, we can create equivalence classes of homotopies. The homotopy equivalence class of a curve is just the set of all curves that are homotopic to a chosen representative. So, given path f, we say that $[f]$ is the equivalence class of all paths g that are homotopic to f.

We can think of homotopies as a family of continuous functions dependent on a parameter, typically t. If we restrict to homeomorphism rather than simply continuous functions, we define what is called as an *isotopy*.

7.3. The Fundamental Group, π_1**.** Let x_0 be some fixed base point in topological space X. Consider two loops f and g based at x_0 and the equivalence classes $[f]$ and $[g]$. Define the operation $[f][g] = [f \circ g]$. This operation is well defined and can be seen by choosing some $f' \in [f]$ and $g' \in [g]$ and showing that $[f'][g'] = [f][g]$.

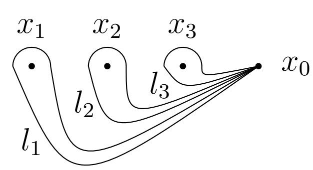


FIGURE 1.3. Here we consider a set of 3 points in the unordered configuration space of $UC_3(\mathbb{R}^2)$. Then we fix a base point x_0. The loops, l_i, are the generators for the fundamental group of $\pi_1(UC_3(\mathbb{R}^2), x_0)$.

Consider the constant map given by $c_i : [0,1] \to [0,1]$ and consider the map $f^{-1} = f(1-t)$. We can see that when composing f with f^{-1}, we return the constant map c_i. We will take these to be our identity, c_i, and our inverse, f^{-1}. We can also clearly see that associativity will hold as $f \circ (g \circ h) \simeq (f \circ g) \circ h$. Therefore we have formed a group! This group is called the fundamental group and is the group of homotopoy classes where the endpoints are fixed throughout the homotopy with operation of composition as seen earlier. Concretely, we define:

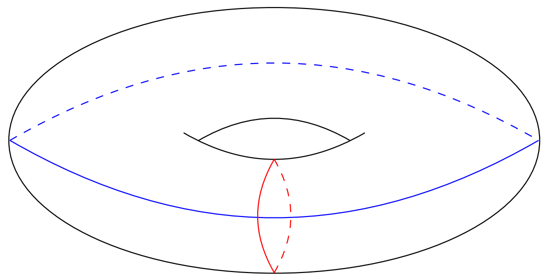
DEFINITION. Let X be a topological space and x_0 be some fixed base point in X. Define the *fundamental group* to be the group $\pi_1(X, x_0)$ with respect to $[f][g] = [f \circ g]$ for loops f, g based at x_0.

EXAMPLE 3. Consider $\pi_1(T^2, (x_0, y_0))$. The torus is defined as $S^1 \times S^1$, so we can write the fundamental group of the torus as $\pi_1(S^1 \times S^1, (x_0, y_0))$ with some base point (x_0, y_0). Thus, $\pi_1(S^1 \times S^1, (x_0, y_0)) \approx$

$\pi_1(S^1, x_0) \times \pi_1(S^1, y_0)$. Since the circle is isomorphic to the additive group of integers, we have that

$$\pi_1(T^2, (x_0, y_0)) \approx \mathbb{Z} \times \mathbb{Z}.$$

We can see this from the following drawing:



We can see that this fundamental group has two generators given by the red circle (a_1) and by the blue (b_1) which have the relation of $a_1 b_1 a_1^{-1} b_1^{-1} = 1$.

Example 4. Consider the fundamental group of the 2-holed torus: $\pi_1(T^2 \# T^2)$. Rather than the two loops as shown for the torus example, this connected sum will have generators of a_1, b_1, a_2, b_2 with the relation of $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1$. This can be generalized to the connected sum of n tori. For an n-holed torus, we have that the fundamental group is generated by $a_1, b_1, \ldots a_n, b_n$ with the relation of $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1} = 1$

CHAPTER 2

Braid Groups

Physical braids can be seen everywhere in life from hair styles to ropes. In the study of braids, mathematical braids become an abstraction of the familiar braided hair or rope. Rather than gathering all the strands of the braid with a pony tail at the end as we do with hair, the ends of the strands of a mathematical braid remain separate and secured to a surface.

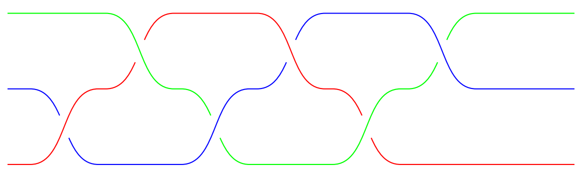


FIGURE 2.1. A 3-stand braid in the familiar braided hair pattern.

To be more concrete, a braid is formed by the crossings of n many number of strands or strings. For any fixed natural number n, the set B_n consists of all n-strand braids with a group structure. So B_n denotes our braid group. We know from group theory that each group must have an identity element. For braid groups, the *identity braid* is a braid with no crossings. For $n = 3$, the identity braid is shown in Fig 2.2.

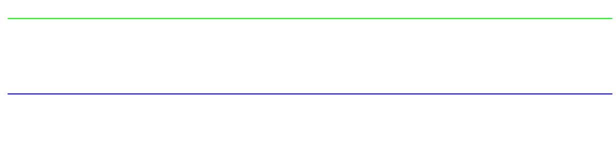


FIGURE 2.2. The identity braid for 3 strands.

The braid group B_n can be expressed in terms of generators and relations in the following way:

Theorem 2.1. *A braid group, B_n, has a presentation give by*

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2\rangle$$

1. Important Subgroups of B_n

1.1. The Pure Braids. Consider the map $\pi : B_n \to \sum_n$ where $\sum_n$ is the n-element permutation group. The *pure braid group*, denoted PB_n consists of all braids in B_n such that, after permutation of the strings, they return to the original order as which they began, i.e. the kernel of the map $\pi : B_n \to \sum_n$. Formally, we say:

Definition. Let $\pi : B_n \to \sum_n$ be the map from the n-strand braid group, B_n, to the n-element permutation group. The *pure braid group*, denoted PB_n, is given by:

$$PB_n = Ker(\pi : B_n \to \sum_n).$$

If we consider B_3, the a typical 3-strand pure braid group in B_3 is shown in Fig. 2.3

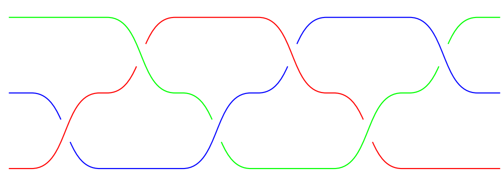


Figure 2.3. A 3-strand pure braid. As we can see, the strands start and end in the same positions. In other words, $(1, 2, 3) \to (1, 2, 3)$.

The "standard" generating set was first described by Artin to be:

$$A_{ij} = (\sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1})^{-1}, \tag{2.1}$$

however each A_{ij} can be usedto show commuting relationships [**1**]. For example, A_{23} comutes with $A_{12}A_{13}$.

1.2. The Full Twist, Δ^2. If we consider a braid with n strands, connected to two walls, then we rotated the right most wall by 2π or 360 degrees. The braid generated by this action is called a full twist, denoted Δ^2, and is shown in Fig 2.4 below.

In fact, the full twist, Δ^2, generates the center of the braid group, B_n. In other words, any braid that commutes with any other braid will be some integer multiple of the full twist [**1**]. This fact implies that for

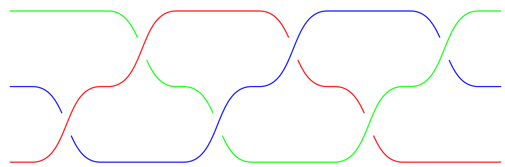


FIGURE 2.4. The full twist, Δ^2

B_m to be isomorphic to B_n the condition that $m = n$ must be satisfied [**10**].

As we can see, braids are incredibly geometric and visual mathematical entities, but we can also express them as the permutation of a set of points.

2. Configuration Spaces

Typically, we define a *configuration space*, also called state or parameter space, as the space that contains all possible states of a given system.

If particles collectively confined to a given region undergo motions, the resulting mapping of the motion over time forms a braid. The starting position of these particles is referred to as the *configuration* of n particles with the unordered set of all possible configurations as:

$$UC_n(\mathbb{R}^2) = \{\{p_1, \ldots, p_n\} \subset \mathbb{R}^2 : p_i \neq p_j \text{ for } i \neq j\}.$$

If we think of a braid as attached to a wall or disk, as we slide that disk along the braid, at each point in time we have a point in the unordered configuration space. Furthermore, a braid can be described by a function $\gamma : [0, 1] \to UC_n(\mathbb{R}^2)$. First, choose a base point $x_0 \in UC_n(M)$ where M is a connected manifold of dimension at least 2. Then choose an arbitrary element $\beta \in \pi_1(UC_n(M), x_0)$ in the fundamental group of the n particle unordered configuration space. In essence, braids can be expressed as loops in the unordered configuration space.

THEOREM 2.2. [**1**] *Let M be a connected manifold of dimension ≥ 2. Then, the fundamental group of $UC_n(M)$ is isomorphic to the braid group $B_n(M)$.*

Consider the real plane, $\mathbb{R}^2$, and 4 points. Say these points lie on the $x - axis$ starting at $(1, 0)$ and increasing by one along the axis until $(4, 0)$, for convenience. The generators of the corresponding fundamental group, $\pi_1(UC_4(\mathbb{R}^2), q_0)$ where q_0 is our base point, are the loops that are based at q_0 and contain each individual point. Therefore, there will be 4 generating loops. The isomorphism between this and a geometric braid arises by assigning a loop in the fundamental group (or composition of loops) to a braid. Since at each point in time, a braid is just comprised of points in an unordered configuration space, we can understand this braid as a loop in this space. In other words, rather than viewing a braid in a 3-dimensional sense, we can compress the braid (put together the right and left walls) and see our loop.

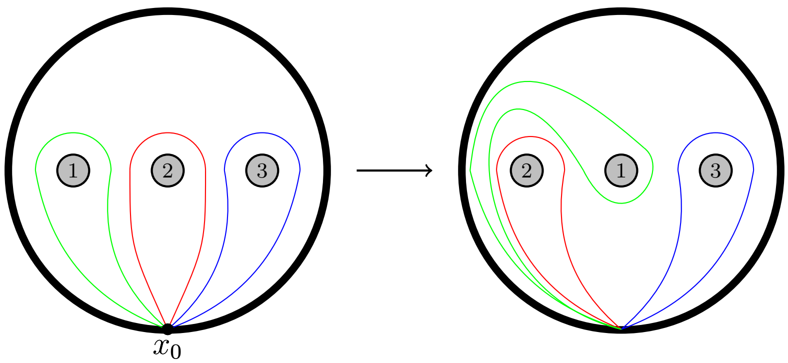


FIGURE 2.5. Depiction of the action on the fundamental group with base point x_0 where punctures 1 and 2 on the disk are exchanged in clockwise fashion. This is isomorphic to the 3 strand braid of σ_1.

Viewing braids in this way (see Figure 2.5) allows us to investigate generalizations from a more topological perspective and relate different groups, such as the mapping class group (this will be described in depth in the next chapter).

3. Braid Groups on Different Surfaces

3.1. Braids on a Sphere, S^2. The braid group on the 2-sphere or S^2 is very similar to the braid group on the disk except the points permute on the 2-sphere rather than the disk. This causes the braid group on the 2-sphere to have the same generators as those of the disk but with an extra relation. We can depict the braids in a similar fashion to that of the disk; however, we can also visualize the braids geometrically between two concentric spheres where the outer most sphere is at $t = 0$ and progresses inward with time (i.e. a continuum of spheres). Consider the 2-sphere with n permutable punctures which are also the ends of the n-braid. Similar to the construction of braids on the disk or plane, we map the 2-sphere to itself allowing for the permutation of punctures. This map creates a continuum of spheres and forms a spherical braid and admits the presentation shown in Equation 2.2.

The braid group on the sphere or the spherical braid group has a presentation of:

$$
\begin{aligned}
B_n(S^2) = \langle \delta_1, \ldots, \delta_{n-1} :&\delta_i\delta_j = \delta_j\delta_i \text{ for } |i-j| > 1, \\
&\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1} \text{ for } 1 \leq i \leq n-2, \\
&\delta_1\delta_2\ldots\delta_{n-1}\delta_{n-1}\ldots\delta_2\delta_1 = 1\rangle
\end{aligned} \tag{2.2}
$$

The generators of the spherical braid group are the same as those of the disk: σ_i denotes the clockwise interchange of the ith and the $i+1$th puncture. For convenience, as we will refer motion on the disk versus the sphere frequently in this chapter, we will denote the motion of punctures on the disk as σ_i and the motion of punctures on the sphere as δ_i although they represent the same type of motion.

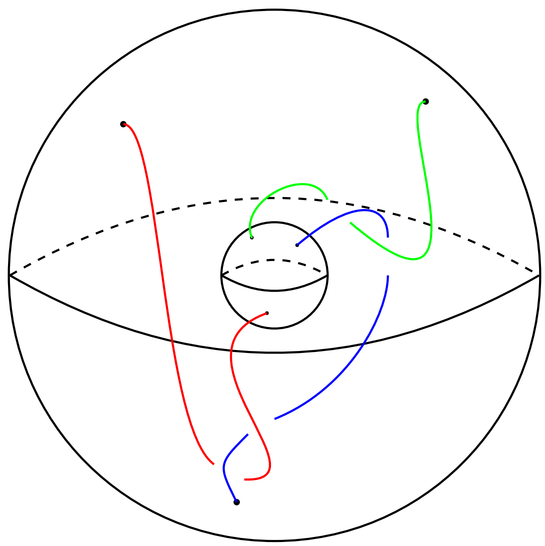


FIGURE 2.6. Example of a non-trivial spherical braid with three strands.

3.2. Torsion in $B_n(S^2)$. Upon first glance, the braid group of the sphere and of the disk are almost identical except for the last extra relation of $\delta_1\delta_2\ldots\delta_{n-1}\delta_{n-1}\ldots\delta_2\delta_1 = 1$. This extra relation allows for this group to have torsion as the element $\delta_1\delta_2\ldots\delta_{n-1}$ has order 2 whereas the braid group of the disk is *torsion-free*.

In the context of the 2-sphere, the full twist has finite order, order 2, within our spherical braid group. The spherical braid group is has *torsion*, i.e. there exists elements of finite order [**4**].

This can be understood from the fact that (Δ^2) has roots of finite order in the spherical braid group [**1**].

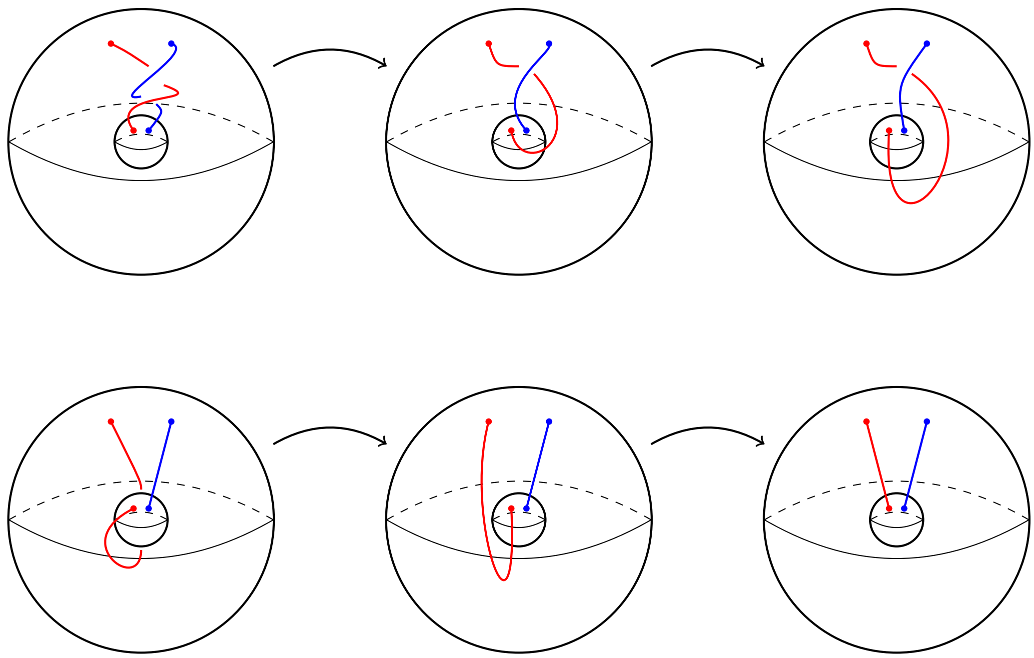


Figure 2.7. Example of how a a seemingly non-trivial spherical braid is, in fact, the identity braid.

3.3. Braids on a Torus, T^2. Braids on a torus are similar to those on the surface of a sphere in that we have a lot more options for how strands can cross. On a torus, we have the same generators, σ_i, but we can also polodially and torodially permute the strands with generators, ρ_j and τ_j respectively. While there are $n-1$ number of possible σ_i generators, there are n possible ρ_j and τ_j. In other words, we have generators σ_i, ρ_j, τ_j for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

We can think of the torus as the space $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ [**3**]. This allows us to clearly see and represent (using matrices) the configuration space. Since a braid will exist in a continuum of nested tori, torus braids are incredibly difficult to visualize geometrically as we've done previously with the braids on a disk and sphere. Since the fundamental group of the configuration space will be isomorphic to the corresponding braid group, we often times investigate π_1 when the braid is incredibly difficult to visualize and manipulate like the torus braids. Moreover, the relationship between the fundamental group and mapping class groups allows us to make important conclusions about these types of maps.

CHAPTER 3

Mapping Class Groups

Let S_g be a surface with genus g. Then the mapping class group is defined as

$$Mod(S_g) = \pi_0(Homeo^+(S_g), \partial S_g), \tag{3.1}$$

where $\pi_0(X)$ is the set of path-connected components of topological space X. In other words, the mapping class group of surface S_g is considered to be the group of isotopy classes of elements of $Homeo^+(S_g, \partial S_g)$, where isotopies fix the boundary pointwise [**3**]. Elements of $Mod(S_g)$ are called *mapping classes* and may be denoted $MCG(S)$, $Map(S)$, $\Gamma_{g,n}$, etc, but for the purposes of this thesis we will use $Mod(S_g)$ for the mapping class group of S_g. The notation of $Mod(S_g)$ is meant to be synonomous with "modular group" as it can be viewed as a generalization of the classical modular group $SL(2, \mathbb{Z})$ [**3**].

In the context of motions on different surfaces, we consider S_g to be an oriented topological manifold with genus g and possibly with boundary ∂S_g. If S is a surface with punctures, then we will consider theses punctures as marked points on S such that the mapping class group of S leaves the set of marked points invariant modulo isotopy. Note that the mapping class group allows permutation of punctures and marked points but must pointwise fix the boundary. In other words, isotopies must fix each boundary component pointwise, but can rotate a neighborhood of a puncture [**3**]. Let Q be a finite subset of the interior of our surface. Each isotopy class is defined by the self-homeomorphisms of the pair (S_g, Q) such that homeomorphism $f : S_g \to S_g$ fixes ∂S_g pointwise and Q setwise, but preserves the orientation of Q, i.e. $f(x) = x$ for all $x \in \partial S_g$ and $f(Q) = Q$. Each self-homeomorphism of the pair (S_g, Q) induces a permutation on Q which, when filtered using isotopy, establishes our mapping class group. Note that if $Q = \emptyset$, then $Mod(S_g, Q) = Mod(S_g, \emptyset) = Mod(S_g)$.

1. Examples of Mapping Classes

1.1. The disk and the once-punctured disk. Without punctures, the mapping class group of the closed disk is trivial [**3**]. This is called the Alexander lemma:

LEMMA 3.1. *The group $Mod(D^2)$ is trivial.*

This lemma states that for any homeomorphism, φ, of the disk, there exists an isotopy of φ to the identity through homeomorphisms that are the identity on the boundary [**3**]. The proof for this lemma is detailed in [**3**], but the idea is that D^2 is contractible to a point through what is called as the "Alexander Trick." This same proof also holds for the once punctured disk.

LEMMA 3.2. *The group $Mod(D^2, x_0)$ is trivial.*

1.2. The sphere and the punctured sphere. The other two mapping class groups that are trivial are of the sphere, S^2 or S_0, and the once-punctured sphere denoted $S_{0,1}$, $Mod(S^2)$ and $Mod(S_{0,1})$ respectively. To prove that $Mod(S_{0,1})$ is trivial, we can identify the once-punctured sphere with the real plane, $\mathbb{R}^2$, using the stereographic projection and then use the fact that all orientation-preserving homeomorphisms of $\mathbb{R}^2$ is homotopic to the identity by the straight-line homotopy [**3**]. This example can be modified by isotopy such that it fixes a point in order to show that $Mod(S^2)$ is also trivial.

The mapping class group of the thrice-punctured sphere is more complicated and computed using a fixed arc on the surface of $S_{0,3}$. By cutting along the arc, we create a new surface: the punctured disk. This allows us to be able to apply the Alexander Lemma (Lemma 3.1) and establish a procedure for computing the mapping class group of surfaces [**3**]. First, it is important to note that any two essential simple proper arcs in $S_{0,3}$ with the same endpoints are isotopic and any two essential arcs that both start and end at the same marked point of $S_{0,3}$ are also isotopic [**3**].

Choose an arc α in $S_{0,3}$ with distinct endpoints p and q. Let $\phi : Mod(S_{0,3}) \to \sum_3$ be a homomorphism that fixes three marked points, p, q, r. Since ϕ fixes these marked points, the endpoints of $\phi(\alpha)$ are also p and q. Therefore, $\phi(\alpha)$ is isotopic to α. By cutting along α, we can obtain a disk with one marked point and the boundary is established by α with marked point r. Thus, since ϕ preserves orientations of both our thrice-punctured sphere and α, ϕ induces a homeomorphism, $\bar{\phi}$, of this disk with one marked point which is the identity on the boundary. Then, by Lemma 3.1, the mapping class group of the once-marked disk is trivial and so the induced homeomorphism, $\bar{\phi}$, is homotopic to the identity. Thus, $Mod(S_{0,3})$ is isomorphic to the symmetric group on three letters, $\sum_3$. This description was based on the proof detailed by Farb et al. [**3**].

PROPOSITION 3.3. *The natural map* $Mod(S_{0,3}) \to \sum_3$ *given by the action of* $Mod(S_{0,3})$ *on the set of marked points of* $S_{0,3}$ *is an isomorphism* [**3**].

An analogous method can be applied to the twice-punctured disk to show that there is a homomorphism between $Mod(S_{0,2})$ and $\mathbb{Z}_2$ or $\mathbb{Z}/2\mathbb{Z}$.

1.2.1. *The torus.* The mapping class group of the torus is a well studied and important example of mapping class groups. The importance of the mapping class group of the torus lies in its exhibition of hints to the behavior of higher-genus surfaces [**3**].

THEOREM 3.4. *The homomorphism* $\sigma : Mod(T^2) \to SL(2,\mathbb{Z})$ *given by the action on* $H_1(T;\mathbb{Z}) \approx \mathbb{Z}^2$ *is an isomorphism* [**3**].

1.2.2. *The four-times-punctured sphere.* Recall that we can think of the torus as a square with opposite sides identified. In general, the hyperelliptic involution map is a map from a surface to itself that rotates along an axis which then fixes specific marked points. By applying the hyperelliptic involution map to T^2, ι, the square rotates by angle π about the center of the square (see figure 3.1 and 3.2 for a visual illustration of this map). This map has exactly four fixed points and so the quotient has four distinguished points which is topologically the same as the four-times-punctured sphere.

Therefore, there exists a strong relationship between the mapping class groups of the torus and the four-times-punctured sphere. We can obtain the mapping class group of the four-times-punctured sphere using the hyperelliptic involution map. If a map of the torus commutes with ι, every such element of the mapping class group of the torus induces an element in the mapping class group of $S_{0,4}$. This relationship allows the computation of $Mod(S_{0,4})$.

PROPOSITION 3.5. $Mod(S_{0,4}) \approx PSL(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

The computations of the mapping class groups of the above examples, $S_{0,2}$, $S_{0,3}$, $S_{0,4}$, T^2, all follow the same general algorithm by finding a collection of arcs that, when cut along, make the surface into disks which allows the application of the Alexander Lemma (Lemma 3.1). In other words, the mapping class groups is determined by the action on isotopy classes of these curves and arcs [**3**].

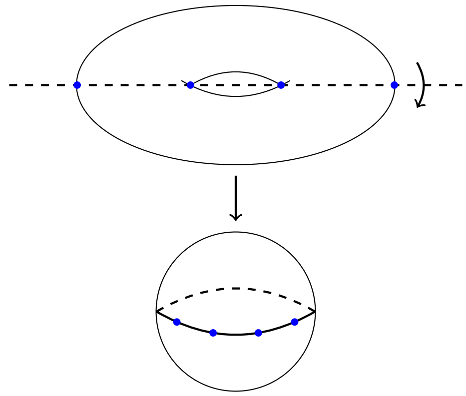


FIGURE 3.1. Depiction of a hyperelliptic involution map. We can see that when the torus is rotated by 180 degrees along the dashed axis, we see 4 marked points (shown in blue). Thus we established a relationship between the 2-sphere with 4 marked points and the torus under a hyperelliptic involution map.

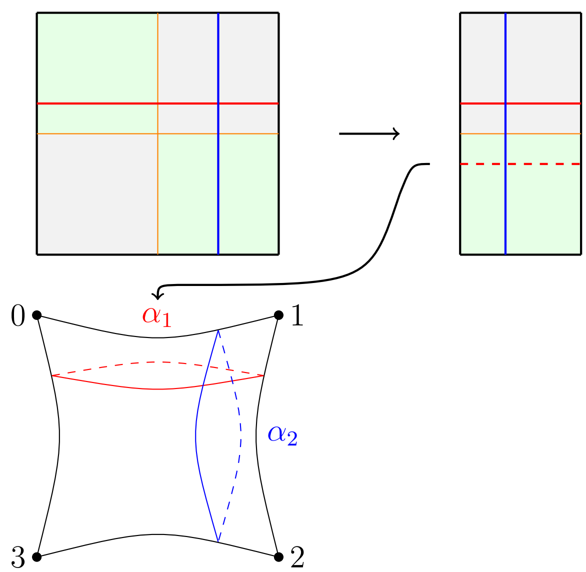


FIGURE 3.2. Another depiction of a hyperelliptic involution map starting with the torus, T^2, with sides identified and resulting in the four times punctured sphere, $S_{0,4}$.

1.3. Birman Exact Sequence. Let S be a compact surface with n marked points in the interior, i.e. $(S, \{x_1, \ldots, x_n\}$ such that each

$x_i \in S^\circ = S - \partial S$. Define the *forgetful map* or *forgetful homeomorphism* $Forget : Mod(S, \{x_1, \ldots, x_n\} \to Mod(S)$ via "forgetting" that the marked points are in fact marked. Now define $Push : \pi_1(UC(S,n) \to Mod(S, \{x_1, \ldots, x_n\}$ by taking a curve in the fundamental group and "pushing" it to an element in the mapping class group.

We can relate the mapping classes through the following theorem.

THEOREM 3.6. ***(Birman Exact Sequence, Generalized).*** *Let S be a surface without marked points and with $\pi_1(Homeo^+(S, \partial S)) = 1$. The following sequence is exact:*

$$1 \to \pi_1(UC(S,n)) \xrightarrow{Push} Mod(S, \{x_1, \ldots, x_n\}) \xrightarrow{Forget} Mod(S) \to 1.$$

where the map $Forget$ simply forgets that the marked points are marked, $Push$ is the map that takes a curve in the fundamental group and pushes it to an element in the mapping class group, and $Homeo^+(S, \partial S)$ is the group of orientation preserving homeomorphisms of S that pointwise fix the boundary and preserve the set of marked points. In addition, $\pi_1(UC(S,n))$ is the fundamental group of the unordered configuration space of the surface S with n marked points.

Upon inspection of the Birman exact sequence, we can see that there might be a relationship between mapping class groups and braid groups since a braid group can be expressed as the fundamental group of an unordered configuration space. However, there is only an isomorphism between B_n and $Mod(D^2, \{x_1, \ldots, x_n\})$.

We can describe the braid group $B_n(D^2)$ as a mapping class group.

PROPOSITION 3.7. *Let D^2_n be a closed disk with n marked points. Then $B_n(D^2) \approx Mod(D^2, n) = \pi_0(Homeo^+(D_n, \partial D_n)$.*

For an orientable surface S of genus g, we have a relationship of a homomorphism between $B_n(S) \to Mod(S_{g,n})$.

For an orientable surface S of genus g, we have a relationship of a homomorphism between $B_n(S) \to Mod(S_{g,n})$. Since braid groups on surfaces that are not the sphere are extremely difficult to visual, this relationship will be exploited later in this thesis in order to examine maps on different surfaces and surfaces of higher genus.

CHAPTER 4

Fluid Flow and Braids on a Disk

1. Fluid Stirring Protocols and Braids

Fluid motion can be expressed as the collective motion of fluid particles. Since particles cannot split, fluid motion has a well-defined future and remains distinguishable for all time. We can therefore introduce a map that describes the overall evolution of a fluid that occupies a given region over time. In general, this map is defined by $f : B_0 \to B_T$ where B_0 and B_T are the regions occupied by the fluid after time 0 and T. This map is called the *time-T fluid map.* We can consider our inputs to the time-T fluid map as vectors or points within the region and denote this with $\mathbf{p}, \mathbf{q}$. So, if $\mathbf{p}, \mathbf{q}$ are distinct then $f(\mathbf{p}) \neq f(\mathbf{q})$ and thus f is an injective function. Then, by definition of fluid flow, we have intuitively that f is also surjective (i.e. for every $\mathbf{p}' \in B_T$, there exists a $\mathbf{p} \in B_0$ such that $f(\mathbf{p}) = \mathbf{p}'$). Thus, we can say that the fluid map is a bijection based on the assumptions of fluids that preserves the number of fluid points. Essentially, the assumptions when modeling fluid motion is that a fluid map is a bijection that preserves the cardinality or number of fluid points with an inverse map such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$ where f is a homeomorphism.

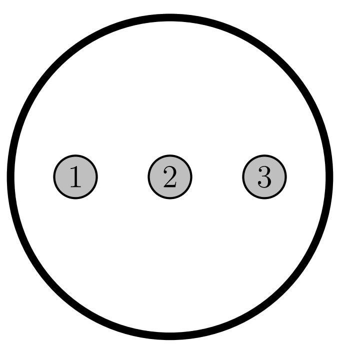


Figure 4.1. Generic stirring device with $n = 3$ rods.

1.1. Stirring Protocol as a Braid. Consider a container (like a cylinder) filled with fluid in which a stirring device consistent of mobile rods is placed. If we take the overhead image of this system, we see a disk with circles. Take this to be a disk with punctures, D_n, where n denotes the number of punctures and therefore rods. In physical space, these rods can move around in disk, but can never intersect each other. If we then create a plot of the horizontal position of the rods over time, we can obtain a plot of the movement of the rods. In a mathematical sense, the resulting graph is a tangle of strands that cannot intersect each other, and therefore this graph describes a braid.

In a more rigorous sense, consider the disk that is filled with fluid. The domain is the disk with n punctures where each puncture represents a rod and the disk is filled with fluid. The stirring action is the permutation of these punctures over time where the domain is mapped to itself. We assume that this mapping is a bijection and the inverse is differentiable, i.e. this map is a diffeomorphism [**6**]. During the interchange of the rods, each fluid particle moves from an initial position to a final one causing the mixing or stirring of the fluid (the movement of fluid particles).

Let $f : D_n \to D_n$ be the stirring motion corresponding to n rods. We say that f is isotopic to the identity if there exists a parametrized set of fixed rod diffeomorphisms, $\psi_\tau, 0 \leq \tau \leq 1$, such that $\psi_0 = \mathrm{id}$ and $\psi_1 = f$ [**6**]. For more that 3 rods, there exists maps that are not isotopic to the identity. The important conclusion is that stirring motions that are not isotopic to the identity results in the effect called "topological chaos," which is essentially that the motions of the rods become unreplicable and describes the overall complexity that cannot be removed by continuous deformations of the shape of the region [**6**]. Topological chaos is derived from the topological properties of the stirring motion. This motion, as described previously, is simply the permutation of rod positions and can thus define a physical braid of n strands over time. Formally, pick n points on the disk. Then as time progresses, we can think of the disk moving along a straight line with respect to time allowing for the permutation of points on the disk and fixing the boundary pointwise. Then after some time, a braid appears as a collection of n non-intersecting strands that connect to each of the specific points (see Figure 4.2).

Braids then specify isotopy classes on the n punctured disk, and in general: a configuration space with n punctures [**6**]. The generators of the braid group, as described in depth in Chapter 2, are σ_i where $i = 1, 2, \ldots, n$ and each σ_i represents the interchange of the ith and $(i+1)$th strands/rods with inverses σ_i^{-1}. In general, the group has the

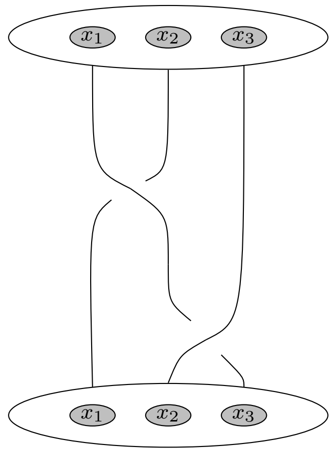


FIGURE 4.2. Depiction of how a mathematical braid arises from a stirring protocol. In this figure, we show the 3-strand braid word $\sigma_1\sigma_2^{-1}$.

presentation:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2\rangle.$$

The braid group and the isotopy classes serves as a "label" for equivalent stirring motions on a surface (in this previous explanation, we consider the disk, but we will also investigate the implications on different surfaces).

2. Boyland Matrix Representation of Braids

Consider the case of a 3-strand braid on the disk. Along with a more pictorial depiction of braids, we can also represent 3-strand braids using 2×2 matrices with integer entries. Another way to understand braids is the deformation of material lines that connect each puncture as shown in Figure 4.3. First, the particles are connected by material lines and then are transformed into the second picture in Figure 4.3 under braid motion σ_2.

Since line I is mapped to I' and in essence II to itself, we can represent σ_2 (shown in Figure 4.3) with the matrix

$$M_{\sigma_2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.1}$$

where the rows of the matrix correspond to the original lines and the columns are the transformed lines. Since I' is comprised of I and II, we place a 1 in both entries of the first column whereas II' consists only

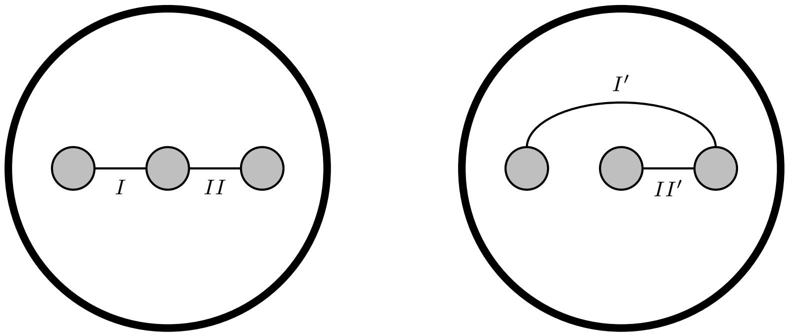


FIGURE 4.3. The transformation of lines connecting punctures from initial position I and II to I' and II' up to homotopy ([**6**]).

of II so the first entry of the second column has a 0 and the second a 1. Similarly, the matrix representing σ_1 is

$$M_{\sigma_1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

Since σ_1^{-1} is the inverse of σ_1, we can construct $M_{\sigma_1^{-1}}$ by taking the inverse of M_{σ_1}.

$$M_{\sigma_1^{-1}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

In summary, the generators σ_1 and σ_2 can be expressed in their matrix representation as

$$[\sigma_1] = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{4.4}$$

Thus if we consider the braid $\beta = \sigma_1^{-1}\sigma_2$, the matrix corresponding to this braid is

$$M_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.5}$$

3. Thurston-Nielsen Classification Theorem

Thurston-Nielsen theory categorizes diffeomorphisms in terms of isotopy classes [**6**]. Each isotopy class contains a *Thurston-Nielsen representative* which is the "simplest" in the isotopy class and once understood, presents the topological complexity in each diffeomrophism of the isotopy class. Below is the classification theorem as given in Boyland et al.'s paper in the precise mathematical language [**6**]:

THEOREM 4.1. ***Thurston-Nielsen Classification Theorem:*** *if f is a homeomorphism of a compact surface, S, then f is isotopic to a homeomorphism, φ, of one of the following types:*

(i) *Finite order: $\varphi^n = id$ for some integer $n > 0$.*
(ii) *Pseudo-Anosov: φ preserves a pair of transverse, measured foliations, $\mathcal{F}_u$ and $\mathcal{F}_s$, and there is a $\lambda > 1$ such that φ stretches $\mathcal{F}_u$ by a factor of λ and contracts $\mathcal{F}_s$ by $1/\lambda$.*
(iii) *Reducible: φ fixes a family of reducing curves, and on the complementary surfaces φ satisfies (i) or (ii).*

This theorem states that the Thurston-Nielsen (TN) representative, φ, is either finite order, pseudo-Anosov, or reducible. Finite order homeomorphisms are the simpliest as the composition of the TN representative with itself a finite number of times is the identity. The second type is the more interesting type and is much more complicated. In brief, a pseudo-Anosov map is a generalization of linear maps from a manifold to itself with marked local directions of expansion (with a stretching factor of λ) and contractions. Since pseudo-Anosov maps have these marked areas of contraction and expansion, it makes them an appealing map to study when analyzing and optimizing stirring protocols.

From a fluid stirring perspective, pseudo-Anosov diffeomorphisms are particularly intriguing as they give area-preserving maps of these regions of uniform stretching or contracting at each given point meaning that there is no elliptic island where fluid gets trapped and not mixed [**6**]. Boyland et al. found that there are more advantages to a stirring protocol of three or more stirrers as topological chaos can be built into the respective protocol. To do this, they employ braid groups (as diffeomorphisms) and compute the topological entropy of the corresponding map (discussed in detail in Chapter 6). For the case of 3 or more strands or stirrers, the Thurston-Nielson classification theory states that a diffeomorphism (a specific stirring protocol) will either be isotopic to finite order, reducible, or pseudo-Anosov. Therefore, we can choose the diffeomorphisms that are pseudo-Anosov to begin our investigation. The stretching factor or dilitation of a pseudo-Anosov map, λ, can be found by representing each diffeomrophism using matrices with integer entries and finding the spectral radius. The natural log of this factor, $\log \lambda$, is the elusive topological entropy and describes the overall complexity of the system and dictates how good a stirring protocol can be.

CHAPTER 5

Topological Entropy

In thermodynamics, entropy is defined as the natural logarithm of the number of possible microstates all multiplied by Boltzmann's constant. Furthermore, entropy describes the disorder of a system. In the context of a dynamical system, topological entropy is the same: a number to understand the total disorder of a system. For fluid motion, we can understand the mixing capabilities of a set of rods using this entity. Topological entropy, however useful a number to calculate, is also extremely computationally expensive to actually compute.

In 2000, Boyland et al. described an approach to show how particular stirring protocols increase system complexity–topological chaos [**6**]. This method classifies stirring protocols as braids diffeomorphisms and utilizes the Thurston-Nielsen classification theorem (theorem 4.1) in order to draw conclusions from pseudo-Anosov maps.

First, we will give a rigorous definition of topological entropy and then in the following sections, will give a brief explanation of the various strategies on how to estimate and understand topological entropy using mathematical techniques. In order to do so, we must first define Anosov and pseudo-Anosov (pA) maps.

1. Anosov and Pseudo-Anosov Maps

In brief, a pA map is simply a generalization of Anosov maps with marked directions of expansion and contraction. In order to describe a pA map, we must first discuss an Anosov map.

Linear Anosov diffeomorphisms are maps on the two-dimensional torus T^2. The theory begins with a matrix with unit determinant and trace strictly greater than 2, i.e.:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } \det(M) = 1 \text{ and } \mathrm{Tr}(M) > 2. \tag{5.1}$$

Furthermore, the matrix M ensures that it has two distinct real eigenvalues of $\lambda > 1$ and $1/\lambda$. These eigenvalues mark expansion and contraction on the torus where the eigendirection corresponding to λ is the unstable direction and the eigendirection corresponding to $1/\lambda$

is the stable. The collection of all the unstable directions establish the unstable foliation. Thus, the collection of all stable directions defines the stable foliation. If a diffeomorphism, f, is isotopic to an Anosov map, then interating this diffeomorphism grows the lengths of loops at least at the rate of λ^n [**6**]. There also exists periodic points and thus the number of fixed points of f^n grows again at a rate of at least λ^n.

The pseudo-Anosov case is similar to the Anosov case except defined on different surfaces and cannot posses a non-vanishing vector field as on the two-torus [**6**]. However, pA maps still contain uniform expansion and contractions by factor λ. Formally, a homeomorphism $\varphi : F \to F$ of a surface F of negative Euler characteristic is *pseudo-Anosov* if there exists a transverse pair of measured foliations on the surface F with stable $(\mathcal{F}^s, \mu^s))$ and unstable $(\mathcal{F}^u, \mu^u)$ foliations together with a representative f of φ such that $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $f(\mathcal{F}^s) = \mathcal{F}^s/\lambda$ [**16**]. We call λ the *dilitation* or stretch factor of φ. The pA map, ϕ, still shares most of the attributes of an Anosov map: the number of fixed points of φ^n grows like λ^n, iterated loops converge to the unstable foliation.

In terms of fluid stirring, pA diffeomorphisms give area-preserving maps of a region with uniform expansion and contraction thus prohibiting elliptic islands which impede mixing [**6**]. This consequence makes pA stirring protocols especially attractive and the subject of investigation. A way of describing the complexity of a stirring protocol is by calculating the topological entropy.

2. Topological Entropy of a Map

We can think of the topological entropy of a map as the measure of the complexity of a dynamical system. The topological entropy of a map f is given by $h_{top}(f)$ which is dependent on the growth rate of a sequence (a_n) [**7**]. The growth rate of a sequence is defined as

$$\text{Growth}_{n\to\infty} a_n = \max\left\{1, \lim_{n\to\infty} \sup |a_n|^{1/n}\right\}. \tag{5.2}$$

We say that if the growth rate of the sequence is strictly greater than 1 (i.e. when $\text{Growth}_{n\to\infty} a_n > 1$), then the sequence grows *exponentially* which yields the desired result of topological chaos [**12**]. Typically, if a sequence grows exponentially with growth rate λ, then the sequence will grow like λ^n as n tends to infinity [**7**]. We now define topological entropy as:

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log(a_n). \tag{5.3}$$

2.1. Strict Definition of Topological Entropy. If a sequence grows exponentially, say by $\log(\lambda)$, will grow iteratively by λ^n as n tends to infinity. So we can also define topological entropy of a map by $\text{Fix}^\infty(f) = \text{growth}(\#(\text{Fix}(f^n))$. If $\text{Fix}(f^n))$ is infinite, then our entropy definition is not computable. Therefore, we can replace $\text{Fix}(f^n)$ with the Nielsen number.

2.2. Nielsen Equivalence. Recall that given a compact space X and mapping $f : X \to X$, the number of fixed points is the cardinality of the set $\text{Fix}f = \{x \in X : f(x) = x\}$. Given a map $f : X \to X$ such that $\text{Fix}f \neq \emptyset$, we say that two fixed points $x, y \in \text{Fix}f$ are *Nielsen equivalent* if there exists a path $P : [0, 1] \to X$ such that $P(0) = x$ and $P(1) = y$ where $f \circ P \sim P$ relative to endpoints x, y. In other words, the path P can be continuously deformed to $f(C)$ keeping x and y fixed.

The Nielsen number of f is defined as $N(f) = \#\{\mathscr{F} : I(f, \mathscr{F}) \neq 0\}$, where $I(f, \mathscr{F})$ is the fixed point index of f at the fixed point class $\mathscr{F}$. We can say that the Nielsen number is the number of essential (fixed point index is not equal to zero) Nielsen classes. This number is defined to be a non-negative integer and as it contains at least one fixed point of f, the inequality $0 \leq N(f) \leq \#\text{Fix}f$ is true.

2.3. Definition Topological Entropy using Nielsen Classes. In other words, topological entropy is the exponential growth rate of period orbits. Since $\text{Fix}(f^n)$ is not always finite, we can consider the growth rate of period Nielsen classes, $\text{pnt}(f, n)$. For each $n \in \mathbb{N}$, we define period Nielsen classes as the number of distinct period n-periodic Nielsen classes for f with $\text{pnt}^\infty(f) = \text{growth}(\text{pnt}(f, n))$ which will always be finite [**7**]. Since $\text{pnt}^\infty(f)$ is finite, we can understand the topological entropy using the period Nielson classes as a lower bound for topological entropy of a map using Theorem 5.1 [**7**].

THEOREM 5.1. *Given a homeomorphism $f : M \to M$, then $h_{top}(f) \geq pnt^\infty(f)$.*

We can also think as topological entropy as the total exponential complexity of the orbit structure represented by a single number [**11**].

The definition of topological entropy is based on work by Katok and Hasselblatt [**11**] and Boyland et al [**7**]. But before defining and understanding topological entropy, recall the definition of a metric space: a metric space is a set equipped with a function, typically denoted d (the metric) which takes every x, y pair satisfying:

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x,y) = d(y,x)$
3. $d(x,y) + d(y,z) \geq d(x,z)$ (the triangle inequality)

Then note that a compact metric space X is considered compact if each open cover of our space has a finite subcover.

We begin our investigation into the topological entropy by considering a self continuous map between to compact metric spaces: so, consider $f : X \to X$ with a distance function d. We can then define a sequence d_n^f for $n \in \mathbb{N}$ by [**11**]:

$$d_n^f(x,y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) \tag{5.4}$$

This increasing sequence of metrics measures the distance between $\{x, \dots f^{n-1}x\}$ and $\{y, \dots, f^{n-1}y\}$. If we consider the growth rate of the minimal number of initial conditions whose behavior up to time n approximates the behavior of any initial condition up to ϵ, we can define this growth as

$$h_d(f,\epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log(S_d(f,\epsilon,n)) \tag{5.5}$$

where $S_d(f,\epsilon,n)$ is the sequence of initial conditions that approximate the behavior of f, to time n, up to ϵ. Formally, we can write an (n,ϵ)-spanning set (we can denote this set $E \subset X$) as the union of open sets or open balls in X. In other words, $E = \bigcup_{x\in E} B_f(x,\epsilon,n)$ where the open ball in X is $B_f(x,\epsilon,n) = \{x \in X | d_n^f(x,y) < \epsilon\}$. Then $S_d(f,\epsilon,n)$ is just the minimal cardinality of the spanning set. Since equation 5.5 does not decrease with ϵ, we can finally formally define *topological entropy of a map* as

$$h_{top}(f) = \lim_{\epsilon\to 0} h_d(f,\epsilon). \tag{5.6}$$

2.4. Surface Diffeomorphisms. [**15**], [**16**]

A homeomorphism $\varphi : F \to F$ of a surface F of negative Euler characteristic is *pseudo-Anosov* if there exists a transverse pair of measured foliations on the surface F with stable $(\mathscr{F}^s, \mu^s))$ and unstable $(\mathscr{F}^u, \mu^u)$ foliations together with a representative f of φ such that $f(\mathscr{F}^u) = \lambda\mathscr{F}^u$ and $f(\mathscr{F}^s) = \mathscr{F}^s/\lambda$ [**16**]. We call λ the *dilitation* or stretch factor of φ.

Consider M to be a compact oriented surface and denote $\mathscr{J}(M)$ as the homotopy class of closed and simply connected paths not homotopic to zero or the boundary. If $\alpha \in \mathscr{J}$ and $(\mathscr{F},\mu)$ is a measured foliation of M, we can say then that $\mathscr{J}(\mathscr{F},\mu,\alpha) = \inf_{\gamma\in\alpha} \int_\gamma |\mu|$. For any two $\alpha, \beta \in \mathscr{J}(M)$, we can denote the minimum number of intersections by $c(\alpha, \beta)$. The following proposition suggests a method to beginning the computation of topological entropy of a pseudo-Anosov map of a compact oriented surface M [**15**]:

PROPOSITION 5.2. *Let φ be a pseudo-Anosov map with stable and unstable foliations $(\mathscr{F}^s, \mu^s))$ and $(\mathscr{F}^u, \mu^u)$. Let λ be a real number, $\lambda > 1$, be such that $\varphi(\mathscr{F}^s) = (1/\lambda)\mathscr{F}^s$ and $\varphi(\mathscr{F}^u) = \lambda \mathscr{F}^u$. Then:*

(1) $\lim_{n\to\infty} \frac{c(\varphi^n \alpha, \beta)}{\lambda^n} = \mathscr{I}(\mathscr{F}^s, \mu^s, \alpha)\mathscr{I}(\mathscr{F}^u, \mu^u, \alpha)$,

(2) $h_{top}(\varphi) = \log \lambda$

(3) $h_{top}(\varphi) = \inf\{h_{top}(\psi), \psi \in \mathit{Diff}(M), \psi \sim \varphi\}$.

2.5. Method One: Counting Intersections. Following Proposition 5.2, pick $\alpha, \beta \in \mathscr{I}(M)$ and find the logarithm of the minimum number of intersections between the m-iterate homeomorphism map of α (i.e. $\varphi^m \alpha$) and β:

$$\lim_{m\to\infty} \frac{1}{m} \log c(\varphi^m \alpha, \beta) \tag{5.7}$$

Furthermore, if we only consider surfaces that are homeomorphic to the sphere with finite n discs removed represented with a stereographic projection with respect to one of the removed discs as the pole, we can count the intersections of two sets of curves up to homotopy. The first set L is consistent of n-disjoint simple closed curves whereas the second set R is comprised of n simple closed curves on a sphere with n open discs removed [**15**].

PROPOSITION 5.3. *Let M be a sphere with $n \geq 3$ discs removed, let φ be a diffeomorphism of M and let L and R be the two subsets of $\mathscr{I}(M)$. Then*

$$h_{top}(\varphi) = \lim_{m\to\infty} \tfrac{1}{m} \log c(\varphi^m L, R).$$

See [**15**] for the proof of the above proposition.

3. Topological Entropy of a Braid

As seen in Chapter 1, the braid group of n strands has the group presentation of

$$\begin{aligned} B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2 \rangle. \end{aligned}$$

However, there are many ways to represent the braid group. One useful definition is using the connection between the braid group and surface diffeomorphisms. We can simply think of a braid as a smooth mapping of the permutations of n-distinct points on the disk. In other words, we can represent the braid groups with a diffeomorphism of these points over time with the union of the boundaries of the disk up to homotopy

$$B_n = \mathrm{Diff}(D^2, rel\Gamma_n, \partial D^2)/\sim . \tag{5.8}$$

where D^2 represents the unit disc with n disjoint open discs removed, Γ_n the union of their boundaries, ∂D^2 [**15**].

Consider the configuration space of m distinct points $d_1, \ldots, d_m$ on a disk with base point $c = \{d_1, \ldots, d_2\}$. We denote this configuration spaces as $C_{n,m}(D^2)$. Then, the braid groups on the n-punctured disk is defined as [**12**]:

$$B_n = \pi_1(UC(D^2), n). \tag{5.9}$$

Considering $\beta \in B_n$ to be an n-strand braid, the topological entropy of β is defined as:

$$h(\beta) = \inf_{\varphi\in\beta} h_{top}(\varphi). \tag{5.10}$$

As described earlier, one way of estimating topological entropy of a map is by counting the number of intersections between the m-iterate homeomorphism map of α (i.e. $\varphi^m\alpha$) and β as done in Proposition 5.3. We can extend this idea of counting intersections of closed and simply connected paths to integral laminations. An integral lamination is a set of disjoint non homotopic simple closed curves of a compact and oriented surface [**15**] which are considered up to homotopy. We will denote the set of integral laminations of a surface M as $\mathscr{L}(M)$.

PROPOSITION 5.4. [**15**] *Let n be an integer, $n \geq 2$, $L \in \mathscr{L}(M_n)$ and $c(L)$ denote the minimum number of intersections between L and the real axis. if $\rho(L) = (a_1, b_1, \ldots, a_n, b_n)$, then*

$$c(L) = \sum_{i=1}^{n} |b_i| + \sum_{i=1}^{n-1} |a_{i+1} - a_i| + |a_1| + |a_n| + \nu_1/2 + \nu_n/2.$$

This leads to the method for estimation of a braid's entropy [**15**]:

- Choose $\epsilon > 0$, let n be an integer greater than or equal to 1 and $\beta \in B_n(D^2)$.
- First, write the braid using the standard generators, σ_i.

- Compute $\rho(\beta^m L_0^n)$ for $m = 1, 2, 3\ldots$ and $c_n = \frac{1}{m}\log c(\beta^m L_0^n)$ using Dynnikov's formulae, Proposition 5.4 omitting $\nu_1/2 + \nu_n/2$ as they do not change
- Stop when $|c_{m+1} - c_m| < \epsilon$.

CONJECTURE 5.5. [**15**] *Let n be an integer, $n \geq 2$. There exists a positive constant $C_n \in \mathbb{R}$ such that for any braid $\beta \in B_n$ and its corresponding sequence $(c_m)_{m\geq 0}$,*

$$|c_m - h_{top}(\beta)| \leq C_n \frac{\log m}{m}.$$

CONJECTURE 5.6. [**15**] *Braids of maximal entropy belong to B_3 or B_4.*

3.1. Triangulation and Integral Laminations. Moussafir ([**15**]) described a method for estimating the topological entropy of pseudo-Anosov braids based on counting the crossings of integral laminations. In work done by Finn and Thiffeault, Moussafir's method is extended to compute a precise estimate for the topological entropy of braids on the torus using a triangulation of the surface using the evolution of integral laminations [**9**].

A *torus braid* is a braid with generators $\{\sigma_i, \rho_i, \tau_i\}$ where σ_i represents the clockwise interchange of the i-th and $(i+1)$th puncture in the typical braid generation, ρ_i is the i-th puncture making a full 2π rotation poloidally (vertical periodic direction), and τ_i is the i-th puncture making a full 2π rotation toroidally (horizontal period direction). An integral lamination is simply an equivalence class of simple closed curves that are not isotopic to the boundary or any section of the boundary. Finn and Thiffeault describe how these integral laminations are encoded by the triangulation of the flow's domain and describe the details of the evolution of flow under braiding motions [**9**]. The braid's topological entropy is derived from this evolution of integral laminations. By using a specific triangulation of the domain and assuming that the closed curves are pulled tight, counting the number of crossings of the loop and edge of the triangulation is made easier which ultimately gives rise to an estimation for topological entropy given by [**9**]:

$$h_{top} = \log S(n) - \log S(n-1) \tag{5.11}$$

where $S(n)$ is defined as the total number of crossings after n iterations of the braid. Provided that the braid has a pseudo-Anosov component from the Thurston-Nielsen Classification Theorem, the topological entropy estimate described in equation 5.11 requires very few iterations of the braid applications to acquire a precise estimation of the actual topological entropy [**9**]. The natural logarithm of the dilitation (i.e. $\log \lambda$) will always produce a lower bound for the topological entropy; however, the approach described by Finn and Thiffeault provides an accurate and precise estimate of the entropy of a braid described as the permutation of a set of points which can represent the evolution of fluid flow under certain stirring protocols [**9**].

CHAPTER 6

Fluid Stirring on the Sphere, S^2

1. The Sphere, S^2

In mathematics, the term "sphere" can refer to a verity of surfaces. The 0-sphere refers to two points in space, the 1-sphere is the unit circle, and the 2-sphere, the subject of this chapter, refers to the unit ball in three dimensional space. In general, the n-sphere consists of a set of points that are some distance r from a designated point in $n+1$ Euclidean dimensional space. Since topologically, an n-sphere with some arbitrary radius is homeomorphic (topologically the same as) the unit n-sphere, we can just consider the definition of the unit n-sphere. So, the unit circle is the 1-sphere and the globe is the 2-sphere. A sphere that exists in higher dimensional space is called a "hypersphere." For this thesis, we consider the 2-sphere defined as

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^2 : |x|^2 + |y|^2 + |z|^2 = 1\right\}. \tag{6.1}$$

2. Fluid on a Sphere and the Disk

In previous sections, we have described fluid motion as it occurs on the disk; however, in this section, we will extend the methods used for computing and estimating topological etropy for stirring protocols on the disk to those on the sphere.

2.1. Braids on the Sphere, $B_n(S^2)$. Consider the 2-sphere with n punctures. Recall that the group presentation for braids on the sphere is:

$$\begin{aligned} B_n(S^2) = \langle \delta_1, \ldots, \delta_{n-1} :& \delta_i\delta_j = \delta_j\delta_i \text{ for } |i-j| > 1, \\ & \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1} \text{ for } 1 \leq i \leq n-2, \\ & \delta_1\delta_2 \ldots \delta_{n-1}\delta_{n-1} \ldots \delta_2\delta_1 = 1\rangle \end{aligned} \tag{6.2}$$

The last extra relation of $\delta_1\delta_2 \ldots \delta_{n-1}\delta_{n-1} \ldots \delta_2\delta_1$ allows for this group to have elements with finite order. This fundamental difference between spherical braids and braids on a disk prohibits a natural translation of fluid stirring on the disk to that of the sphere; however there does still exist a natural homomorphism $\varphi : B_n(D^2) \to B_n(S^2)$ such that $\varphi(\sigma_i) = \delta_i$.

Although spherical braids have group presentation and are relatively well known entities, they are extremely difficult to visualize. Mapping class groups offer a better avenue of exploration as we can use the Birman short exact sequence in order to relate groups as well as create more pseudo-Anosov maps.

2.1.1. *Mapping Class Group of the Sphere.* Mapping class groups allow us to draw connections and relate groups for ease of computation. Recall that a braid group can be thought of as the fundamental group of the unordered configuration space and that punctures can be thought of as "marked points."

Let $S_{0,n}$ be the sphere with $n = |\{x_0, x_1, \ldots, x_{n-1}\}|$ marked points. Consider the fundamental group $\pi_1(UC(S_0, n)) \approx B_n(S^2)$. We have a point-pushing map $\pi_1(UC(S_0, n)) \to Mod(S_0, \{x_0, x_1, \ldots, x_{n-1}\})$ whose kernel is isomorphic to the image of $\pi_1(Homeo^+(S_0))$ in $\pi_1(UC(S_0, n))$ [**3**]. However, we also know that the group $\pi_1(Homeo^+(S_0)) \approx \mathbb{Z}/2\mathbb{Z}$ and thus when $n \geq 2$ this group maps nontrivially into $\pi_1(UC(S_0, n))$. Furthermore, since $Mod(S_0) = 1$, we have the short exact sequence given in equation 6.3 [**3**].

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(UC(S_0, n)) \to Mod(S_{0,n}) \to 1. \tag{6.3}$$

This short exact sequence provides an explanation as to why there exist trivial elements in the spherical braid group. If we consider the image of some element, $\alpha \in Mod(S_{0,n})$, we can see that this image is in fact a Dehn Twist about a simple closed cure that encloses the entirety of all the punctures, which is in fact a trivial mapping class [**3**]. Moreover, by rotating the n marked points by a 4π twist, we can see that the spherical braid α^2 can be unraveled and shown to be the identity [**3**].

3. Creating pseudo-Anosov maps on punctured spheres.

Using a recipe created by Yvon Verberne, we can establish the construction of pseudo-Anosov maps on n-times punctured spheres using Dehn Twists and positive half twists [**19**]. Let $S_{0,n}$ be the n-times punctured sphere and $Mod(S_{0,n})$ be the corresponding mapping class group. By the Thurston-Nielsen classification theorem, we know that these elements of $Mod(S_{0,n})$ will either be periodic, reducible, or pseudo-Anosov.

3.1. Dehn Twists. In order to continue and use Verberne's construction of pseudo-Anosov maps, we will first explain in brief a Dehn Twist.

First consider the annulus: $A = S^1 \times [0,1]$ and a simple closed curve γ. We say that the Dehn twist about γ is cutting along γ twisting one side of the annulus 2π and then gluing the annulus back together along γ. This method is also called "Dehn twist surgery." In order to perform this surgery on other surfaces S, we define the same region A and define a homeomorphism $\tau : A \to A$ via $(\theta, t) \mapsto (\theta + 2\pi t, t)$. Then we can define another homeomorphism $\varphi : A \to T$ where $T \subset S$ is a cylindrical neighborhood of our smiple closed curve γ. Then, the Dehn Twist about γ is defined as follows:

Definition. Let $D_\gamma^2 : S \to S$ define the *Dehn twist about closed curve* γ via

$$x \mapsto \begin{cases} \varphi \circ \tau \circ \varphi^{-1}(x) & x \in T \\ x & x \notin T \end{cases} .$$

We can illustrate a basic example in Figure 6.1:

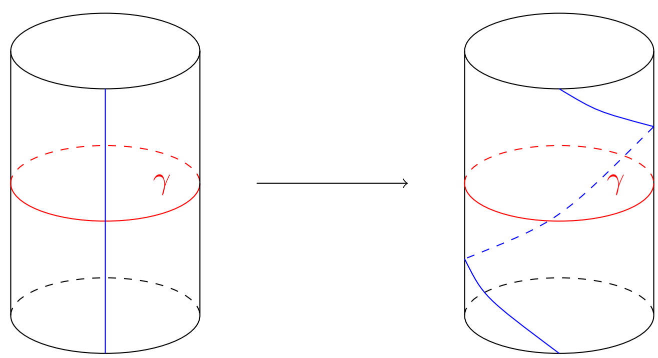


Figure 6.1. Dehn Twist about red curve γ. The blue curve is the resulting image after preforming the twist. The above drawing can also be seen as the T neighborhood on surface S.

3.2. Verberne's Recipe. This recipe is based from both Verberne's thesis (see [**19**]) and the paper published from the thesis (see [**20**]). Let α_j be a simple closed curve that separates punctures j and $j-1$ modulo n. We will define the full Dehn twist around α_j as D_j^2. The corresponding "half twist" is the clockwise interchange of punctures around the corresponding closed curve α_j, denoted D_j. Two half-twists around α_j becomes a full Dehn twist, D_j^2.

The punctures of our sphere are partitioned into sets of evenly spaced punctures. The idea of *evenly spaced* comes from the map $\rho : \mathbb{Z}_n \to \mathbb{Z}_n$ defined via $j \mapsto j+1 \pmod{n}$. A subset of our partition, μ_i, will consist of evenly space punctures if $\rho(\mu_i) = \mu_{i+1}$ for $1 \leq i \leq k$. For example, for the 4-times punctured sphere an example partition set would be $\mu = \{\{0,2\},\{1,3\}\} = \{\mu_1, \mu_2\}$. We now have all the ingredients in order to create our pseudo-Anosov map using Verberne's theorem:

THEOREM 6.1. *Consider the surface $S_{0,n}$. Let $\{\mu_i\}_{i=1}^k$ for $1 < k < n$ be an evenly spaced partition of the punctures of $S_{0,n}$. Then*

$$\phi = \prod_{i=1}^{k} D_{\mu_j}^{q_i} = D_{\mu_k}^{q_k} \dots D_{\mu_2}^{q_2} D_{\mu_1}^{q_1},$$

where $q_j = \{q_{j_1}, \dots q_{j_1}\}$ are tuples of integers greater than one, is a pseudo-Anosov mapping class.

EXAMPLE 5. Consider the 4-times punctured sphere, $S_{0,4}$ and the partition $\mu = \{\{0,2\},\{1,3\}\} = \{\mu_1, \mu_2\}$. Notice that Theorem 6.1 explains that the map $\phi = D_3^2 D_1^2 D_2^2 D_0^2$ is a pseudo-Anosov mapping class.

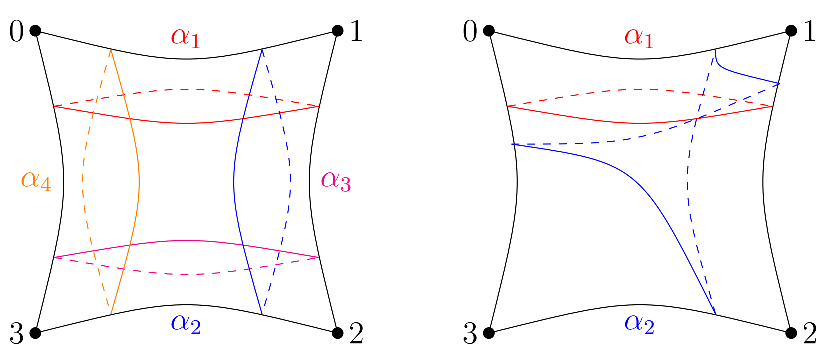


FIGURE 6.2. Depiction of Dehn Twist D_1^2.

4. Hyperelliptic Involution Map, $q : T^2 \to S_{0,4}$.

We can relate the actions on the torus to those on the 4-times punctured sphere using the hyperelliptic involution map. We begin by recalling that the quotient of the torus via equating points through reflection is homeomorphic to the sphere with four singularities [**13**]. We can regard the torus as $\mathbb{R}^2/\mathbb{Z}^2$ and associate points through the mapping $S(x,y) = (2x_0 - x, 2y_0 - y)$ for $(x,y) \in \mathbb{R}^2$. The set of singularities becomes $\{(x_0, y_0), (x_0 + \frac{1}{2}, y_0), (x_0, y_0 + \frac{1}{2}), (x_0 + \frac{1}{2}, y_0 + \frac{1}{2})\}$. We can regard these singularities as punctures and create $S_{0,4}$. Visually, this mapping is the same as considering the torus as a square with sides identified. Folding the square in half is the same as equating points through reflection and S. We can then zip together the identified sides to return the 4-times punctured sphere. This procedure creates the map $q : T^2 \to S_{0,4}$ otherwise known as the hyperelliptic involution map.

4.1. Dehn Twists and the Hyperelliptic Involution Map. Using q as described above, we can create a commutative diagram of mappings as shown in Figure 6.3.

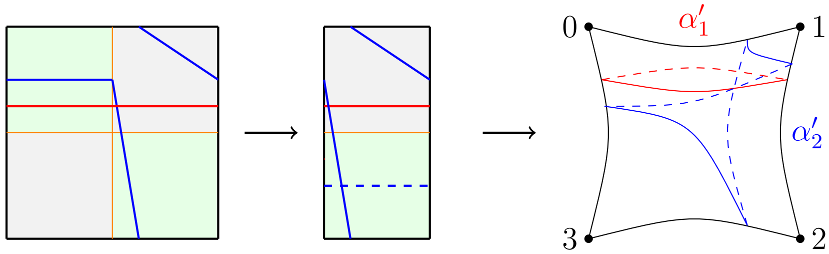
$$\begin{tikzcd}
T^2 \arrow[r, "\widetilde{D}"] \arrow[d, "q"'] & T^2 \arrow[d, "q"] \\
S_{0,4} \arrow[r, "D"'] & S_{0,4}
\end{tikzcd}$$

FIGURE 6.3. Commutative diagram up to homotopy of functions between the torus, T^2 and the 4-times punctured sphere $S_{0,4}$.

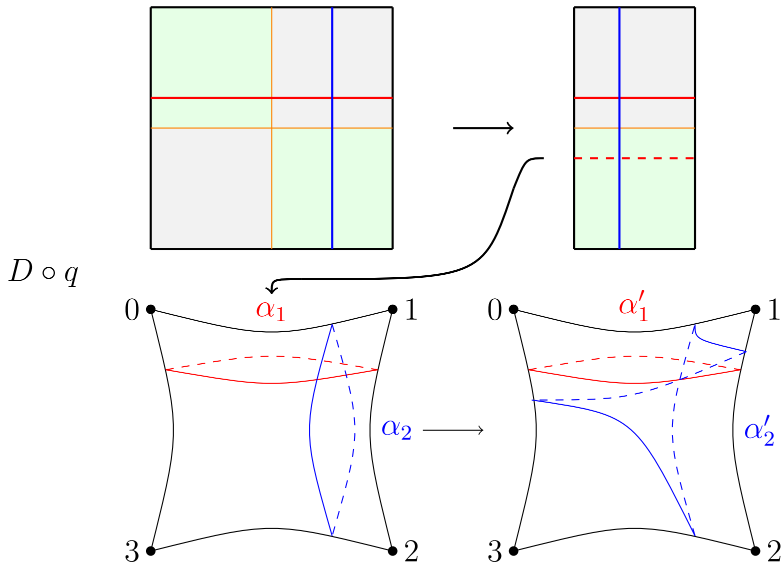
If we consider a map $D : S_{0,4} \to S_{0,4}$ given by Dehn twist D_1^2, we can lift this map to the torus to give the map $\widetilde{D} : T^2 \to T^2$. Composing q with D will yield the same map as composing $\widetilde{D}$ with q, i.e. $\widetilde{D} \circ q = q \circ D$. Since we can express orientation preserving self homeomorphisms the torus in terms of 2×2 matrices in the special linear group with integer matrices, $SL(2, \mathbb{Z})$, we can explicitly calculate the topological entropy of these pseudo-Anosov maps on the 4-times punctured sphere using such matrices. In fact, every pseudo-Anosov map on $S_{0,4}$ can be lifted to an Anosov map on the torus [**13**].

We can show commutative pictorally by the following. First consider $q \circ \widetilde{D}$ as

$q \circ \widetilde{D}$



Now consider $D \circ q$ as



Thus, we can see that the above are equal as $D \circ q = q \circ \widetilde{D}$.

4.2. Topological entropy of a pseudo-Anosov map on $S_{0,4}$. The Dehn twist depicted in Figure 6.1 can be described as the matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the other generator matrix is $M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We can see that D_1^2 is isotopic to D_3^2 and D_0^2 is isotopic to D_2^2. Similarly, $\widetilde{D}_1^2$ is isotopic to $\widetilde{D}_3^2$ and $\widetilde{D}_0^2$ is isotopic to $\widetilde{D}_2^2$. Therefore we can use the same matrix to express similar maps, i.e. matrix M_1 can express both $\widetilde{D}_1^2$ and $\widetilde{D}_3^2$ whereas M_2 represents $\widetilde{D}_0^2$ and $\widetilde{D}_2^2$.

Given a pseudo-Anosov map $f : S_{0,4} \to S_{0,4}$, the topological entropy is $h_{top} = \log(\lambda)$ where λ is the spectral radius of the lifted map $\widetilde{f} : T^2 \to T^2$ [**13**]. This yields an upper bound for the minimal entropy in the isotopy class of the pseudo-Anosov map f [**7**].

Example 6. We return to the same set up as in Example 5: the 4-times punctured sphere. Recall that our pseudo-Anosov map was $\phi = D_3^2 D_1^2 D_2^2 D_0^2$. Using the hyperelliptic involution map, we can lift ϕ to $\widetilde{\phi}$. This map is also pseudo-Anosov [**13**]. Since $\widetilde{D}_0^2$ is homotopic to $\widetilde{D}_2^2$ and $\widetilde{D}_1^2$ is homotopic to $\widetilde{D}_3^2$, we can write

$$\widetilde{\phi} = \left(\widetilde{D}_1^2\right)^2 \left(\widetilde{D}_0^2\right)^2 . \tag{6.4}$$

Furthermore, since the above equation corresponds to matrices, we finally arrive at

$$\widetilde{\phi} = M_1^2 M_2^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}. \tag{6.5}$$

The spectral radius of $\widetilde{\phi}$ is $3 + 2\sqrt{2}$. The topological entropy of $\widetilde{\phi}$ is given by $h_{top}(\widetilde{\phi}) = \log(3 + 2\sqrt{2}) = 1.76275$. This gives us an upper bound for the minimal entropy of the isotopy classes of $\widetilde{\phi}$.

Example 7. Consider the same partition of $\mu = \{\{0, 2\}, \{1, 3\}\}$. We can have another pseudo-Anosov map of

$$\phi_2 = D_3^4 D_1^4 D_2^2 D_0^2 \simeq \left(D_1^2\right)^4 \left(D_0^2\right)^2 \simeq M_1^4 M_2^2 = \begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}.$$

The above has a spectral radius of $5+2\sqrt{6}$. Therefore, we can compute the topological entropy by $h_{top}(\phi_2) = \log(5 + 2\sqrt{6}) = 2.2924..$ This is in fact an upper bound for the minimal entropy of the isotopy class of ϕ_2.

We can see that in both examples, we can produce a pseudo-Anosov map that will have chaotic behavior and have optimal stirring.

CHAPTER 7

Future Work

The main focus of this thesis was to investigate what stirring protocols would look like on the surface of the sphere and how to compute or estimate the topological entropy of these types of maps. One question that still remains if this lifted map of the Dehn twist, $\widetilde{D}$, pseudo-Anosov in general and if not, what are the algebraic conditions which restrict our number of maps.

The homological criterion can described as a condition that gives rise to pseudo-Anosov maps. As worded by Margalit and Spallone in [**14**], the homological criterion is as follows:

PROPOSITION 7.1. *(Homological criterion). Let S be a closed surface of genus at least 2. Let $f \in Mod(S)$, define $\Psi : Mod(S) \to Sp(2g, \mathbb{Z})$, and let $q_f(x)$ be the characteristic polynomial for $\Psi(f)$. If $q_f(x)$ is symplectically irreducible, is not a cyclotomic polynomial, and is not a polynomial in x^k for $k > 1$, then f is pseudo-Anosov.*

Using a hyperelliptic involution, we can create a similar commutative diagram between $T^2 \# T^2$ and $S_{0,6}$. This relationship might require probing the homological criterion in order to use a matrix representation of lifted Dehn twists. An interesting inquiry is whether or not the homological criterion is satisfied for these lifted maps and if not, this would give rise to pseudo-Anosov maps that are not detected by this criterion.

In addition, many have used fixed point theory in order to investigate and estimate topological entropy. Given a homeomorphism (or pseudo-Anosov map as in this thesis) on a surface with negative euler characteristic, we can form the mapping torus. Kawashima was able to investigate the fundmental group of the mapping torus, Γ, and define a trace like quantity that takes values in the free $\mathbb{Z}\Gamma$-module group modulo the set of conjugacy classes [**12**]. Using concepts from Nielsen equivalence classes, the fundamental group of the mapping torus, and this trace like quantity, Kawashima was able to compute the dilatation with some trace like elements using certain representations of the braid groups.

Fixed point theory offers an avenue of investigation to explore optimal estimations of topological entropy as seen in work down by Kawashima [**12**]. This, in addition to the homological representation, offers an additional path to examine.

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