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THE BNS INVARIANTS OF THE GENERALIZED SOLVABLE BAUMSLAG-SOLITAR GROUPS AND OF THEIR FINITE INDEX SUBGROUPS

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ABSTRACT. We compute the Bieri-Neumann-Strebel invariants Σ^1 for the generalized solvable Baumslag-Solitar groups Γ_n and their finite index subgroups. Using Σ^1, we show that certain finite index subgroups of Γ_n cannot be isomorphic to Γ_k for any k. In addition, we use the BNS-invariants to give a new proof of property R_∞ for the groups Γ_n and their finite index subgroups.

1. Introduction

The Bieri-Neumann-Strebel invariant $\Sigma^1(G)$ [1] of a finitely generated group G is an important object of study in geometric group theory and has many connections to other areas of mathematics, especially with the Thurston norm in low dimensional topology. However, the computation of Σ^1 is very difficult in general and there are only few classes of groups for which Σ^1 is known (see e.g. [7] and the references therein).

A group G is said to have property R_∞ if $R(\varphi)$ is infinite for every automorphism $\varphi \in Aut(G)$. Here, $R(\varphi)$ is the number of twisted conjugacy classes of φ, that is, the number of equivalence classes in G given by the relation $g \sim h \Leftrightarrow zg\varphi(z)^{-1} = h$ for some $z \in G$. Twisted conjugacy classes are important in topological fixed point theory.

Let X be a space with universal covering $\tilde{X}$ and $f : X \to X$ be a homeomorphism with induced automorphism $f_* : \pi_1(X) \to \pi_1(X)$. Then $R(f_*)$ is actually the number of (topological) lifting classes of f in $\tilde{X}$ given by a deck transformation conjugation, which also partitions the fixed points of f in X. This number is an upper bound for the Nielsen number $N(f)$, which is a sharp lower bound for the minimal number of fixed points in the homotopy class $[f]$ and one of the main objects of study in Nielsen Theory (see [6]). For instance in [5], property R_∞ was used to show that for any $n \geq 5$, there exists a n-dimensional nilmanifold M such that every self-homeomorphism $f : M \to M$ is isotopic to be fixed point free.

The motivation for this work is [11] in which J. Taback and P. Wong showed property R_∞ for the generalized solvable Baumslag-Solitar groups Γ_n and for every group quasi-isometric to Γ_n, using geometric group theoretic techniques. In [4], D. Gonçalves and D. Kochloukova used the Bieri-Neumann-Strebel (BNS or Σ^1) invariant to deduce property R_∞ for certain classes of groups, including a new proof of the property R_∞ for the Thompson's group F. Since the

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Σ-invariants of the Baumslag-Solitar groups $BS(1,n)$ are sufficient to guarantee property R_∞, it is natural to ask whether property R_∞ for Γ_n and for their finite index subgroups can also be deduced using Σ^1.

In this paper, we show that the property R_∞ for Γ_n and for their finite index subgroups can be deduced from their respective BNS-invariants. Here we compute the Σ^1 invariants of Γ_n and of all its finite index subgroups H. We show that these invariants lie in an open hemisphere of the corresponding character spheres so that property R_∞ follows from [4]. Furthermore, we extend the result to any finite direct product of these groups. Using Σ^1, we show that there exist finite index sugbroups of Γ_n that cannot be isomorphic to any Γ_k, in contrast to the fact that every finite index subgroup of a solvable Baumslag-Solitar group $BS(1,n)$ is again a $BS(1,k)$.

The paper is organized as follows. In section 2 we compute the Σ^1 for Γ_n (Theorem 2.4). In section 3, we classify all the finite index subgroups H of Γ_n in terms of specific generators and index (Theorem 3.4), and give a presentation of H (Theorem 3.5). Then we compute their Σ^1 invariant (Theorem 3.8) and use it to show that some H cannot be a generalized solvable Baumslag-Solitar group (Theorem 3.9). In section 4 we use geometric arguments about the behavior of the induced homeomorphisms $\varphi^* : S(G) \to S(G)$ to show that finding some special invariant convex polytopes in the character sphere of a finitely generated group G is sufficient to guarantee property R_∞ for G. In section 5, we give new proofs (Theorems 5.2 and 5.3) of property R_∞ for the groups Γ_n and H above and also for any finite direct product of them (Theorem 5.4). Finally, in Proposition 5.6, we exhibit a family of groups G where Theorem 4.8 can be used to guarantee property R_∞ without complete information on $\Sigma^1(G)$.

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2. Computation of $\Sigma^1(\Gamma_n)$

In this section we compute the Σ^1 invariants of the generalized solvable Baumslag-Solitar groups Γ_n. First we recall the definition of the BNS-invariant $\Sigma^1(G)$ of a finitely generated group G. There are other equivalent definitions (see [1] and [9]) but we employ the following for our purposes.

Definition 2.1. Let G be a finitely generated group. The character sphere of G is the quotient space

$$S(G) = (Hom(G,\mathbb{R}) - \{0\})/\sim\, = \{[\chi] \mid \chi \in Hom(G,\mathbb{R}) - \{0\}\},$$

where $\chi \sim \chi' \Leftrightarrow r\chi = \chi'$ for some $r > 0$.

It is well known that if the free rank of the abelianized group G^{ab} is n with generators $x_1, ..., x_n$, then $S(G) \simeq S^{n-1}$ with homeomorphism

$$\mathfrak{h} : S(G) \longrightarrow S^{n-1}$$
$$[\chi] \longmapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\|(\chi(x_1), ..., \chi(x_n))\|}.$$

Following [9], we have

Definition 2.2. Let G be a finitely generated group with finite generating set $X \subset G$. Denote by $\Gamma = \Gamma(G, X)$ the Cayley graph of G with respect to X. The first Σ-invariant (or BNS invariant) of G is

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma_\chi \text{ is connected}\},$$

where Γ_χ is the subgraph of Γ whose vertices are the elements $g \in G$ with $\chi(g) \geq 0$ and whose edges are those of Γ which connect two such vertices.

The solvable Baumslag-Solitar group $BS(1, n), n > 1$ is defined by the presentation

$$BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle.$$

We consider the following solvable generalization of $BS(1, n)$.

Definition 2.3. Let $n \geq 2$ be a positive integer with prime decomposition $n = {p_1}^{y_1}...{p_r}^{y_r}$, the p_i being pairwise distinct. We define the solvable generalization of the Baumslag-Solitar group by

$$\Gamma_n = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{{p_i}^{y_i}},\ i = 1, ..., r \right\rangle.$$

More generally, let $S = \{n_1, ..., n_r\}$ be a set of pairwise coprime positive integers such that $n_i \geq 2$ for some i. Define

$$\Gamma(S) = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{n_i},\ i = 1, ..., r \right\rangle.$$

The group $\Gamma(S)$ is always torsion-free.

Note that $BS(1, n)$ is a metabelian group and it admits the following splitting

$$1 \to \mathbb{Z}\left[\frac{1}{n}\right] \to BS(1, n) \overset{\leftarrow}{\to} \mathbb{Z} \to 1.$$

where $\mathbb{Z}\left[\frac{1}{n}\right]$ denotes the n-adic rationals and contains the commutator subgroup $[BS(1, n), BS(1, n)]$. Similarly, Γ_n is characterized by the following short exact sequence

$$1 \to \mathbb{Z}\left[\frac{1}{n}\right] \to \Gamma_n \overset{\varphi}{\to} \mathbb{Z}^r \to 1. \tag{2.1}$$

Here, φ is the canonical projection with $a \mapsto 1$, $\mathbb{Z}\left[\frac{1}{n}\right] = \langle a_j,\ j \in \mathbb{Z} \mid a_j^n = a_{j+1},\ j \in \mathbb{Z}\rangle$ and is generated by the elements

$$a_j = (t_1...t_r)^j a (t_1...t_r)^{-j} \in \Gamma_n.$$

Using the presentation $\mathbb{Z}^r = \langle t_1, ..., t_r \mid t_i t_j = t_j t_i, i \neq j\rangle$, the exact sequence (2.1) splits using the section $\mathbb{Z}^r \to \Gamma_n$ sending $t_i \mapsto t_i$. Thus, $\Gamma_n = \mathbb{Z}\left[\frac{1}{n}\right] \rtimes \mathbb{Z}^r$ is the semidirect product of these two subgroups, and every element $w \in \Gamma_n$ can be uniquely written as $w = t_1^{\alpha_1}...t_r^{\alpha_r}u$ for $u \in \mathbb{Z}\left[\frac{1}{n}\right]$ and $\alpha_i \in \mathbb{Z}$ (we put u on the right side following the notation from Bogopolski in [3]). Observe that the "t_i-coordinates" in Γ_n are well behaved, that is, $(t_1^{\alpha_1}...t_r^{\alpha_r}u)(t_1^{\beta_1}...t_r^{\beta_r}u') = t_1^{\alpha_1+\beta_1}...t_r^{\alpha_r+\beta_r}u''$ for some $u'' \in \mathbb{Z}\left[\frac{1}{n}\right]$. Secondly, because of the presentation of the subgroup $\mathbb{Z}\left[\frac{1}{n}\right]$, we see that any two generators a_i, a_j must be powers of the common generator $a_{\min\{i,j\}}$. Note that $\mathbb{Z}\left[\frac{1}{n}\right]$ is an infinitely generated abelian group and Γ_n is metabelian.

Theorem 2.4. *The complement $\Sigma^1(\Gamma(S))^c$ of the Σ^1 of the group*

$$\Gamma(S) = \left\langle a, t_1, ..., t_r \mid t_it_j = t_jt_i,\ i \neq j,\ t_iat_i{}^{-1} = a^{n_i},\ i = 1, ..., r \right\rangle$$

is given by

$$\Sigma^1(\Gamma(S))^c = \{[\chi_i] \mid \chi_i(t_i) = 1 \textit{and } \chi_i(t_j) = 0 \textit{ for } j \neq i\},$$

In particular, if $n = p_1^{y_1}...p_r^{y_r}$ is a prime decomposition, then

$$\Sigma^1(\Gamma_n)^c = \{[\chi_1], ..., [\chi_r]\}.$$

Furthermore, $\Sigma^1(\Gamma(S))^c$ lies inside an open hemisphere in $S(\Gamma(S))$.

Proof. As pointed out in [1], the Σ^1 coincides with $\Sigma_{G'}$ of [2]. For the metabelian group $\Gamma(S)$, the quotient $\mathbb{Z}^r$ is the torsion-free part of the abelianization so that $S(\Gamma(S)) = S(\mathbb{Z}^r)$. It follows from Proposition 2.1 and formula (2.3) of [2] that

$$\Sigma^1(\Gamma(S)) = \bigcup_{\lambda \in C(A)} \{[\chi] \in S(\Gamma(S)) \mid \chi(\lambda) > 0\}$$

where $A = Ker\varphi = \mathbb{Z}\left[\frac{1}{n}\right]$ as a $\mathbb{Z}[\mathbb{Z}^r]$-module and $C(A) = \{\lambda \in \mathbb{Z}[\mathbb{Z}^r] \mid \lambda \cdot \alpha = \alpha, \text{for all } \alpha \in A\}$ is the centralizer of A. Let $[\chi] \in S(\Gamma(S))$.

Case (1): If $\chi(t_i) < 0$ for some $i, 1 \leq i \leq r$ then we let $\lambda = n_it_i^{-1}$. Note that $n_it_i^{-1} \cdot a = t_i^{-1}a^{n_i}t_i = a$. It follows that $\lambda \in C(A)$ and $[\chi] \in \Sigma^1(\Gamma(S))$.

Let $I_k = \{i_j \mid 1 \leq i_1 < ... < i_k \leq r\}$, $k \geq 2$, be a subset of the set $I = \{1, 2, ..., r\}$.

Case (2): If $\chi(t_{i_j}) > 0$ for $i_j \in I_k$ and $\chi(s) = 0$ for $s \in I \setminus I_k$ then we let $\lambda = \sum_{j=1}^k \alpha_jt_{i_j}$ where α_j are integers such that $\alpha_1n_1 + ... + \alpha_rn_r = 1$ since $n_{i_1}, ..., n_{i_r}$ are pairwise relatively prime. It is easy to see that $\lambda \in C(A)$.

If $\chi(\lambda) > 0$ then $[\chi] \in \Sigma^1(\Gamma(S))$.

Now suppose $\kappa = \chi(\lambda) \leq 0$. Without loss of generality, we may assume that $\alpha_{i_1} > 0$. Since $n_{i_1}t_{i_1}^{-1} \in C(A)$, it follows that for any integer M, $M\lambda - (M-1)(n_{i_1}t_{i_1}^{-1}) \cdot a = a^{M-(M-1)} = a$ so that $\hat{\lambda}_M = M\lambda - (M-1)(n_{i_1}t_{i_1}^{-1}) \in C(A)$. Now, it is straightforward to see that $\chi(\hat{\lambda}_M) = (M-1)n_{i_1}\chi(t_{i_1}) + M\kappa$. There exists a positive integer M such that $\chi(\hat{\lambda}_M) > 0$. In other words, $[\chi] \in \Sigma^1(\Gamma(S))$.

Now, the set of characters that do not belong to Case (1) or Case (2) is $\{[\chi_i]\}$, where $\chi_i(t_i) = 1$ and $\chi_i(t_j) = 0$ if $j \neq i$. To see that this set is the complement of $\Sigma^1(\Gamma(S))$, it suffices to show that $[\chi_i] \in \Sigma^1(\Gamma(S))^c$ for each i. Observe that if $\gamma = \sum c_jt_j^{q_j} \in C(A)$ then either all $q_j > 0$ when $c_j \neq 0$ or for some j, $c_j = n_j$ and $q_j = -1$ with $q_i = 0$ for $i \neq j$. Thus, $\chi_i(\gamma) = c_iq_i$ cannot be positive so each $[\chi_i] \notin \Sigma^1(\Gamma(S))$.

□

Remark 2.5. In an earlier version of this paper, Theorem 2.4 was first proved using a general geometric argument [9, Theorem A3.1].

For the remaining of this paper, we focus on the groups Γ_n.

3. Finite index subgroups of Γ_n

In this section we study the finite index subgroups H of Γ_n. First, in Theorem 3.4 we find a specific set of generators for H using a generalization of an argument given by Bogopolski in [3]. We use these generators to compute the index of H in Γ_n. Then, in Theorem 3.5, we give a presentation for H and, in Theorem 3.8, we compute $\Sigma^1(H)$. We end the section by exhibiting finite index subgroups H of Γ_n which are not isomorphic to Γ_k for any $k \geq 2$.

3.1. **Generators, cosets and index.** The following useful lemma has an elementary proof and was used by Bogopolski in [3].

Lemma 3.1. *Let $n, s \geq 1$ be integers. Let m be the biggest positive divisor of s such that* $\gcd(m,n) = 1$. *Then s divides mn^s.*

To facilitate our computation, we aim to find a *good* set of generators of a finite index subgroup of Γ_n. To do so, we need the next two lemmas.

Lemma 3.2 (Replacing j_0 by any j)**.** *Suppose*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}} a_{q_r}^{l_r}, a_{j_0}^{l} \rangle \leq \Gamma_n \tag{3.1}$$

is a subgroup with arbitrary integers $k_{ii}, l > 0$, $k_{ij} \geq 0$ and $q_i, l_i, j_0 \in \mathbb{Z}$. Then, for any chosen $j \in \mathbb{Z}$, we can replace $a_{j_0}^l$ above by a_j^l, up to modifying $l > 0$ by another positive integer (also called l), that is, $H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}} a_{q_r}^{l_r}, a_j^l \rangle$.

Proof. If $j \leq j_0$ we know from the presentation of $\mathbb{Z}\left[\frac{1}{n}\right]$ that a_{j_0} is a positive power of a_j, so $a_{j_0}^l$ is also a positive power of a_j and the lemma is obviously true. Let us treat the case $j > j_0$. Using that $\mathbb{Z}\left[\frac{1}{n}\right]$ is abelian and the relations of Γ_n, we can show that

$$({t_i}^{k_{ii}}...{t_r}^{k_{ir}} a_{q_i}^{l_i})^{m_i} a_{j_0}^{l} ({t_i}^{k_{ii}}...{t_r}^{k_{ir}} a_{q_i}^{l_i})^{-m_i} = a_{j_0}^{l {p_i}^{m_i y_i k_{ii}} ... {p_r}^{m_i y_r k_{ir}}}$$

for every i and every integer $m_i > 0$. Thus we can replace $a_{j_0}^l$ in the expression of H by this element $a_{j_0}^{l {p_i}^{m_i y_i k_{ii}} ... {p_r}^{m_i y_r k_{ir}}}$, that is, we can multiply the power l of a_{j_0} by ${p_i}^{m_i y_i k_{ii}} ... {p_r}^{m_i y_r k_{ir}}$ in (3.1), and since this new power is still positive we can repeat the process recursively. By doing this for $i = 1, ..., r$ we can replace the power l of a_{j_0} in (3.1) by any number of the form

$$l({p_1}^{m_1 y_1 k_{11}} ... {p_r}^{m_1 y_r k_{1r}})({p_2}^{m_2 y_2 k_{22}} ... {p_r}^{m_2 y_2 k_{2r}}) ... ({p_r}^{m_r y_r k_{rr}})$$

for any $m_1, ..., m_r > 0$. By putting together the first primes in the parentheses we rewrite this as

$${p_1}^{m_1 y_1 k_{11}} {p_2}^{m_2 y_2 k_{22}} ... {p_r}^{m_r y_r k_{rr}} l \lambda$$

for some integer $\lambda > 0$ depending on the m_i. In particular, for the integers $m_i = k_{11} ... \widehat{k_{ii}} ... k_{rr}$ we can replace the power l of a_{j_0} by

$${p_1}^{y_1 k} {p_2}^{y_2 k} ... {p_r}^{y_r k} l \lambda = n^k l \lambda,$$

where $k = k_{11}...k_{rr}$. But $a_{j_0}^{n^k l\lambda} = a_{j_0+k}^{l\lambda}$, which is a positive power of a_{j_0+1}. We repeat this process a finite number of times until we reach the index $j > j_0$ we wanted and the lemma is proved. □

Lemma 3.3 (Replacing l by m)**.** *Let*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}} a_{q_r}^{l_r}, a_j^l \rangle \leq \Gamma_n \tag{3.2}$$

be a subgroup with arbitrary integers $k_{ii}, l > 0$, $k_{ij} \geq 0$ and $q_i, l_i, j \in \mathbb{Z}$. Let m be the biggest divisor of l such that $\gcd(m, n) = 1$. Then we can replace a_j^l by a_j^m in the expression above, that is, $H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}} a_{q_r}^{l_r}, a_j^m \rangle$.

Proof. It suffices to show that the inclusions $a_j^l \in \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}} a_{q_r}^{l_r}, a_j^m \rangle$ and $a_j^m \in H$ hold. The first inclusion is straightforward, because l is a multiple of m and so a_j^l is a power of a_j^m. For the second inclusion first observe that by Lemma 3.1, l must divide mn^l and so it must also divide $mn^{lk_{rr}}$. This implies that the number

$$\gamma = \frac{mn^{lk_{rr}} {p_1}^{y_1(k_{11}-1)lk_{rr}} ... {p_{r-1}}^{y_{r-1}(k_{r-1,r-1}-1)lk_{rr}} \prod_{j=1}^{r-1} \prod_{i=j+1}^{r} {p_i}^{y_i k_{ji} k_{rr} l}}{l}$$

is an integer. Let $A_1, ..., A_r$ be the first r generators of H in (3.2), that is, $H = \langle A_1, ..., A_r, a_j^l \rangle$. It is straightforward to show that

$${A_1}^{-lk_{rr}} ... {A_{r-1}}^{-lk_{rr}} {A_r}^{-l} (a_j^l)^\gamma {A_r}^l {A_{r-1}}^{lk_{rr}} ... {A_1}^{lk_{rr}} = a_j^m,$$

then $a_j^m \in H$, as desired. □

Theorem 3.4. *For any Γ_n, the following properties hold.*

1) *Every finite index subgroup H of Γ_n can be written as*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}} a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}} a^{l_2}, ..., {t_r}^{k_{rr}} a^{l_r}, a^m \rangle \qquad (*)$$

for $0 \leq k_{1i}, ..., k_{i-1,i} < k_{ii}$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$.

2) *If H is any subgroup of Γ_n given by the expression $(*)$ for $0 \leq k_{1i}, ..., k_{i-1,i} < k_{ii}$, $l_i \in \mathbb{Z}$ and $m > 0$ such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$, then $T = \{ {t_1}^{\beta_1} ... {t_r}^{\beta_r} a^j \mid 0 \leq \beta_i < k_{ii},\ 0 \leq j < m \}$ is a transversal of H in Γ_n. In particular, the index of H in Γ_n is $k_{11}...k_{rr}m$ and H has finite index in Γ_n.*

Proof. 1) First, since Γ_n is finitely generated and H is finite index, by the Reidemeister-Schreier theorem H must be also finitely generated and we write

$$H = \langle {t_1}^{\alpha_{11}} ... {t_r}^{\alpha_{1r}} v_1, ..., {t_1}^{\alpha_{m1}} ... {t_r}^{\alpha_{mr}} v_m \rangle$$

for $\alpha_{ij} \in \mathbb{Z}$ and $v_i \in \mathbb{Z}\left[\frac{1}{n}\right]$. Note that $m \geq r$. Otherwise, $\varphi(H)$ would be a subgroup of $\mathbb{Z}^r$ with rank $< r$ and then would have infinite index, a contradiction because φ is surjective. With a similar projection argument, we see that there must be at least one i such that $\alpha_{i1} \neq 0$. Let $k_{11} = \gcd_{\alpha_{i1} \neq 0} \{\alpha_{i1}\}$. Since $k_{11} > 0$ is the smallest positive integer combination of the $\alpha_{i1} \neq 0$, we can obtain inside H an element of the form ${t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1$ for some $k_{12},...,k_{1r} \in \mathbb{Z}$ and $u_1 \in \mathbb{Z}\left[\frac{1}{n}\right]$, so we can write

$$H = \langle {t_1}^{\alpha_{11}}...{t_r}^{\alpha_{1r}}v_1, ..., {t_1}^{\alpha_{m1}}...{t_r}^{\alpha_{mr}}v_m, {t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1\rangle. \tag{3.3}$$

Now, since all the nonzero α_{i1} are multiples of k_{11}, say, $\alpha_{i1} = d_i k_{11}$, we can replace ${t_1}^{\alpha_{i1}}...{t_r}^{\alpha_{ir}}v_i$ by $({t_1}^{\alpha_{i1}}...{t_r}^{\alpha_{ir}}v_i)({t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1)^{-d_i} = {t_2}^{\alpha'_{i2}}...{t_r}^{\alpha'_{ir}}v'_i$ in (3.3). Then, after relabeling these new generators, we can write

$$H = \langle {t_2}^{\alpha_{12}}...{t_r}^{\alpha_{1r}}v_1, ..., {t_2}^{\alpha_{m2}}...{t_r}^{\alpha_{mr}}v_m, {t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1\rangle.$$

We added a new generator and "eliminated" all the t_1 coordinates of the first m generators of H. This was the first step. In a similar way, we can do this for all the other $t_2, ..., t_r$ coordinates. After r steps, we added r new generators and eliminated all the $t_1, ..., t_r$ letters from the first m generators from H, so we have

$$H = \langle v_1, ..., v_m, {t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1, {t_2}^{k_{22}}...{t_r}^{k_{2r}}u_2, ..., {t_r}^{k_{rr}}u_r\rangle$$

with $k_{ii} > 0$ and $v_i, u_i \in \mathbb{Z}\left[\frac{1}{n}\right]$. But in $\mathbb{Z}\left[\frac{1}{n}\right]$ we have $\langle v_1, ..., v_m\rangle = \langle u\rangle$ for some $u \in \mathbb{Z}\left[\frac{1}{n}\right]$ and

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1, {t_2}^{k_{22}}...{t_r}^{k_{2r}}u_2, ..., {t_r}^{k_{rr}}u_r, u\rangle \tag{3.4}$$

By manipulating the generators above if necessary, we may suppose that $0 \leq k_{1i}, ..., k_{i-1,i} < k_{ii}$ (they could be also positive if we wanted) in (3.4). Finally, write $u_i = a_{q_i}^{l_i}, u = a_q^l$ for $q_i, q, l_i, l \in \mathbb{Z}$. Then

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_q^l\rangle. \tag{3.5}$$

Let us show that we may assume $l > 0$ above. If $l \neq 0$ then, up to changing a_q^l by $(a_q^l)^{-1} = a_q^{-1}$ if necessary, we are done. If $l = 0$, that is,

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}\rangle, \tag{3.6}$$

we do the following: since $\mathbb{Z}^r$ is abelian, every commutator of elements in H must be in $Ker(\varphi)$ (and obviously in H). At least one of the commutators between the r generators of H in (3.6) must be non-trivial. Otherwise, H would be a finite index abelian subgroup of Γ_n and we would have $\Sigma^1(\Gamma_n) = S(\Gamma_n)$ by using Proposition $B1.11$ in [9], a contradiction to Theorem 2.4. Then let $a_j^{l'}$ ($l' \neq 0$) be a non-trivial commutator between two generators of H. We can add it to 3.6 and up to changing $a_j^{l'}$ by its inverse, we are done.

Our next steps will be eliminating the subindices q_i from the a letters in the generators of (3.5). Fix some $1 \leq i \leq r$. If $q_i \geq 0$, then $a_{q_i}^{l_i}$ is a power of a and we are done by doing this replacement in (3.5). Suppose $q_i < 0$. By Lemma 3.2 we replace q by q_i in (3.5). Now, let m be the biggest divisor of l such that $\gcd(m, n) = 1$. By Lemma 3.3 we can also replace l by m above and obtain

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^m\rangle.$$

Since $\gcd(m, n) = 1$ we also have $\gcd(m, n^{-q_i}) = 1$ and there must be $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}$ such that $\tilde{\alpha}m + \tilde{\beta}n^{-q_i} = 1$. Then for $\alpha = l_i\tilde{\alpha}$ and $\beta = l_i\tilde{\beta}$ we have $\alpha m + \beta n^{-q_i} = l_i$, or

$$l_i - m\alpha = n^{-q_i}\beta.$$

Then, using the relations in Γ_n we have

$$\begin{aligned} H &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i-m\alpha}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{n^{-q_i}\beta}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{\beta}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \end{aligned}$$

and relabeling β by l_i, m by l and q_i by q again we have

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q}^{l}\rangle,$$

that is, we removed the subindex q_i from $a_{q_i}^{l_i}$ in 3.5. If we do this for all i we remove all the subindices and obtain

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a_{q}^{l}\rangle$$

for some $q \in \mathbb{Z}$. We can use Lemma 3.2 to replace q by 0 and we get the desired set of generators for H. To finish, let m (a new one) be the biggest divisor of l such that $\gcd(m, n) = 1$. By Lemma 3.3, we replace a^l by a^m in the expression above. If $H \cap \langle a\rangle = \langle a^m\rangle$, we are done. If not, let $m' = \min\{k \geq 1 \mid a^k \in H\}$. It's easy to see that $H \cap \langle a\rangle = \langle a^{m'}\rangle$. Since $a^m \in H$, m is a multiple of m' and we have $\gcd(m', n) = 1$. Then, by adding $a^{m'}$ to the set of generators of H, the generator a^m can be removed. By relabeling m' by m, we obtain the desired result.

2) Let H be such a subgroup. As shown in item 1), we may suppose that $k_{ij} > 0$ for all i, j. Let us first show that $\Gamma_n = \bigcup_{{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j \in T} H{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j$. Every element of Γ_n is written as ${t_1}^{-\alpha_1}...{t_r}^{-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r}$ for $\alpha_i, \gamma_i \geq 0$ and $l \in \mathbb{Z}$. Since $k_{ij} > 0$ for all i, j, one can show that every coset of Γ_n is of the form $Ha^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r}$ for $l \in \mathbb{Z}$ and $\gamma_i \geq 0$. Now we claim that every such coset can be also written as $H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^{l'}$ for some integer l'. In fact, because $1 = \gcd(m, n) = \gcd(m, {p_1}^{y_1}...{p_r}^{y_r})$, the prime decomposition of m does not involve any of the p_i. Then it is also true that $\gcd(m, {p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}) = 1$. Let k, k' be integers such that $km + k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r} = 1$. Then $l + (-lk)m = (lk'){p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}$ and relabeling $-lk$ by k and lk' by k' we get $l + km = k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}$. Now since $a^m \in H$ we do

$$\begin{aligned} Ha^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} &= H(a^m)^k a^l {t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{l+km}{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}}{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^{k'} \end{aligned}$$

and relabeling k' by l' we showed the claim. To transform this coset into one of the cosets in the theorem, we apply successive algorithms: choose some index i. If $\gamma_i < k_{ii}$ we stop the algorithm. If $\gamma_i \geq k_{ii}$, by manipulating this coset we show that

$$H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l = H{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i - k_{ii}}{t_{i+1}}^{\gamma'_{i+1}}...{t_r}^{\gamma'_r}a^{l'}$$

for some integer l'. If $\gamma_i - k_{ii} < k_{ii}$ we stop the algorithm. If $\gamma_i - k_{ii} \geq k_{ii}$ we do the above again. Then after finite steps our "i-algorithm" shows that

$$Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l = Ht_1{}^{\gamma_1}...t_{i-1}{}^{\gamma_{i-1}}t_i{}^{\beta_i}t_{i+1}{}^{\gamma'_{i+1}}...t_r{}^{\gamma'_r}a^{l'}$$

for some $0 \leq \beta_i < k_{ii}$. Now, starting with the coset $Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l$, we successively apply the "i-algorithm" for $i = 1, 2, ..., r$ and obtain exactly

$$Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l = Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'}$$

for $0 \leq \beta_i < k_{ii}$ and $l' \in \mathbb{Z}$. Finally, write $l' = qm + j$ for $0 \leq j < m$. Then $Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'} = Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j$ because

$$\begin{aligned} t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'}(t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j)^{-1} &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'-j}t_r{}^{-\beta_r}...t_1{}^{-\beta_1} \\ &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{mq}t_r{}^{-\beta_r}...t_1{}^{-\beta_1} \\ &= (a^m)^{qp_1^{\beta_1 y_1}...p_r{}^{\beta_r y_r}} \in H. \end{aligned}$$

This shows that $\Gamma_n = \bigcup_{t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j \in T} Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j$.

Now let us show that the cosets over T are all distinct. Let $Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j = Ht_1{}^{\beta'_1}...t_r{}^{\beta'_r}a^{j'}$ for $0 \leq \beta_i, \beta'_i < k_{ii}$ and $0 \leq j, j' < m$. By definition,

$$\begin{aligned} w = a^{p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j-j')}t_1{}^{\beta_1-\beta'_1}...t_r{}^{\beta_r-\beta'_r} &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{j-j'}t_1{}^{-\beta_1}...t_r{}^{-\beta_r}t_1{}^{\beta_1-\beta'_1}...t_r{}^{\beta_r-\beta'_r} \\ &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j(t_1{}^{\beta'_1}...t_r{}^{\beta'_r}a^{j'})^{-1} \in H. \end{aligned}$$

Then, projecting in $\mathbb{Z}^r$,

$$(\beta_1 - \beta'_1, ..., \beta_r - \beta'_r) = \varphi(w) \in \varphi(H) = \langle (k_{11}, k_{12}, ..., k_{1r}), (0, k_{22}, ..., k_{2r}), ..., (0, ..., 0, k_{rr}) \rangle .$$

Write

$$(\beta_1 - \beta'_1, ..., \beta_r - \beta'_r) = \lambda_1(k_{11}, k_{12}, ..., k_{1r}) + \lambda_2(0, k_{22}, ..., k_{2r}) + ... + \lambda_r(0, ..., 0, k_{rr})$$

for integers λ_i. Since the first vector $(k_{11}, k_{12}, ..., k_{1r})$ is the only one with non-vanishing first coordinate we have $\beta_1 - \beta'_1 = \lambda_1 k_{11}$. Since $0 \leq \beta_1, \beta'_1 < k_{11}$ we must have $\beta_1 = \beta'_1$ and therefore $\lambda_1 = 0$. By easy induction we can show that all the λ_i must vanish. Now, we just have to show that $j = j'$. We already have $a^{p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j-j')} \in H$. Since $H \cap \langle a \rangle = \langle a^m \rangle$ (by item 1)), we have

$$p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j - j') = qm$$

for some $q \in \mathbb{Z}$. So m divides $p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j - j')$. Since $\gcd(n, m) = 1$, m does not contain any of the p_i in its prime decomposition, and therefore m must divide $j - j'$. Since $0 \leq j, j' < m$ we have $j = j'$, as desired. This completes the proof. $\square$

3.2. **A presentation.** We now give a presentation for an arbitrary finite index subgroup H of Γ_n.

Theorem 3.5. *Let H be any finite index subgroup of Γ_n (see Theorem 3.4), say,*

$$H = \langle {t_1}^{k_{11}} ... {t_r}^{k_{1r}} a^{l_1}, {t_2}^{k_{22}} ... {t_r}^{k_{2r}} a^{l_2}, ..., {t_r}^{k_{rr}} a^{l_r}, a^m \rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$. Then H has the following presentation:

$$H \simeq \left\langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i},\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \right\rangle,$$

where $P_i = p_i^{y_i k_{ii}} ... p_r^{y_r k_{ir}}$ $(i = 1, ..., r)$ and $R_{ij} \in \mathbb{Z}$ characterized by

$$l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m.$$

Proof. It is easy to see that $(t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i}) a^m (t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i})^{-1} = a^{mP_i}$ in Γ_n, for $i = 1, ..., r$. Also, since

$$(t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i})(t_j^{k_{jj}} ... t_r^{k_{jr}} a^{l_j})(t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i})^{-1}(t_j^{k_{jj}} ... t_r^{k_{jr}} a^{l_j})^{-1} = a^{l_i P_i (1-P_j) - l_j P_j (1-P_i)} \in H \cap \langle a \rangle = \langle a^m \rangle,$$

we have $l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m$ for some integer R_{ij}.

We write $(t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i})(t_j^{k_{jj}} ... t_r^{k_{jr}} a^{l_j})(t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i})^{-1}(t_j^{k_{jj}} ... t_r^{k_{jr}} a^{l_j})^{-1} = a^{mR_{ij}}$. Now define a group

$$G = \left\langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i},\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \right\rangle.$$

The group G has the relations

$$x_i \alpha = \alpha^{P_i} x_i,\ x_i \alpha^{-1} = \alpha^{-P_i} x_i,\ x_i x_j = \alpha^{R_{ij}} x_j x_i,\ x_i x_j^{-1} = x_j^{-1} \alpha^{-R_{ij}} x_i,$$

which shows that, for every fixed i, all the x_i-letters in a word with positive power can be pushed right as much as we want. Similarly, the relations

$$\alpha x_i^{-1} = x_i^{-1} \alpha^{P_i},\ \alpha^{-1} x_i^{-1} = x_i^{-1} \alpha^{-P_i},\ x_j x_i^{-1} = x_i^{-1} \alpha^{R_{ij}} x_j,\ x_j^{-1} x_i^{-1} = x_i^{-1} x_j^{-1} \alpha^{-R_{ij}}$$

show that all the x_i-letters in a word with negative power can be pushed left as much as we want. Because of this, any element of G is of the form $x_1^{-\lambda_1} ... x_r^{-\lambda_r} \alpha^M x_r^{\delta_r} ... x_1^{\delta_1}$ for $\lambda_i, \delta_i \geq 0$ and $M \in \mathbb{Z}$. Now let us show that $G \simeq H$. Define $\theta : G \to \Gamma_n$ by putting $\theta(\alpha) = a^m$ and $\theta(x_i) = t_i^{k_{ii}} ... t_r^{k_{ir}} a^{l_i}$ for $i = 1, ..., r$. It is easy to check that θ is a group homomorphism and surjective, so we only need to show that θ is also injective. Indeed, let $w = x_1^{-\lambda_1} ... x_r^{-\lambda_r} \alpha^M x_r^{\delta_r} ... x_1^{\delta_1} \in G$ such that $\theta(w) = 1$. Then

$$\left(t_1^{k_{11}} ... t_r^{k_{1r}} a^{l_1}\right)^{-\lambda_1} ... \left(t_r^{k_{rr}} a^{l_r}\right)^{-\lambda_r} a^{mM} \left(t_r^{k_{rr}} a^{l_r}\right)^{\delta_r} ... \left(t_1^{k_{11}} ... t_r^{k_{1r}} a^{l_1}\right)^{\delta_1} = 1.$$

By projecting both sides of equation above on the t_1-coordinate by the homomorphism $w \mapsto (w)^{t_1}$, we get $k_{11}(\delta_1 - \lambda_1) = 0$ and so $\delta_1 = \lambda_1$. Then by conjugating the above equation on both sides by $(t_1^{k_{11}} ... t_r^{k_{1r}} a^{l_1})^{\lambda_1}$ we get

$$\left(t_2^{k_{22}} ... t_r^{k_{2r}} a^{l_2}\right)^{-\lambda_2} ... \left(t_r^{k_{rr}} a^{l_r}\right)^{-\lambda_r} a^{mM} \left(t_r^{k_{rr}} a^{l_r}\right)^{\delta_r} ... \left(t_2^{k_{22}} ... t_r^{k_{2r}} a^{l_2}\right)^{\delta_2} = 1.$$

By doing this recursively we get $\delta_i = \lambda_i$ for $i = 1, ..., r$ and $a^{mM} = 1$. Then $M = 0$ (since a is torsion free and $m > 0$). Thus $w = x_1^{-\lambda_1} ... x_r^{-\lambda_r} \alpha^0 x_r^{\lambda_r} ... x_1^{\lambda_1} = 1$, as desired. This completes the proof. $\square$

3.3. **The Σ^1 invariant.** Let H be a finite index subgroup of Γ_n, say,

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^m\rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a\rangle = \langle a^m\rangle$. By Theorem 3.5, we write H as

$$H = \left\langle \alpha, x_1, ..., x_r \mid x_i\alpha x_i^{-1} = \alpha^{P_i},\ x_ix_jx_i^{-1}x_j^{-1} = \alpha^{R_{ij}} \right\rangle,$$

for $P_i = p_i^{y_ik_{ii}}...p_r^{y_rk_{ir}}$ $(i = 1, ..., r)$ and some $R_{ij} \in \mathbb{Z}$. Here, $\alpha = a^m$ and $x_i = t_i^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i}$. Since all the $p_i^{y_i}$ are ≥ 2, obviously the P_i also are ≥ 2 and so it is easy to see that α must have torsion in the abelianized group H^{ab}. The x_i are torsion-free, though. So we have the homeomorphism

$$\begin{aligned} \mathfrak{h} : S(H) &\longrightarrow S^{r-1} \\ [\chi] &\longmapsto \frac{(\chi(x_1), ..., \chi(x_r))}{\|(\chi(x_1), ..., \chi(x_r))\|}. \end{aligned}$$

To compute $\Sigma^1(H)$ inside this sphere, we will use the following fact.

Proposition 3.6. *Let G be a finitely generated group and $H \leq G$ a finite index subgroup with inclusion $i : H \to G$ and induced map $i^* : S(G) \to S(H)$, $i^*[\chi] = [\chi \circ i] = [\chi|_H]$. Suppose that any homomorphism $\chi : H \to \mathbb{R}$ can be extended to a homomorphism $\hat{\chi} : G \to \mathbb{R}$. Then*

$$\Sigma^1(H) = i^*(\Sigma^1(G)) \text{ and } \Sigma^1(H)^c = i^*(\Sigma^1(G)^c).$$

Proof. By Proposition $B1.11$ in [9], for any $[\chi] \in S(G)$ we have $[\chi] \in \Sigma^1(G) \Leftrightarrow [\chi|_H] \in \Sigma^1(H)$. Then $i^*(\Sigma^1(G)) \subset \Sigma^1(H)$. On the other hand, let $[\chi] \in \Sigma^1(H)$ and let $\hat{\chi} : G \to \mathbb{R}$ be an extension of χ. We have $[\hat{\chi}|_H] = [\chi] \in \Sigma^1(H)$, so again by Proposition $B1.11$ in [9] we have $[\hat{\chi}] \in \Sigma^1(G)$. Then $[\chi] = i^*[\hat{\chi}] \in i^*(\Sigma^1(G))$, as desired. The other equality is similar. □

Lemma 3.7. *Let H be a finite index subgroup of Γ_n, say,*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^m\rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a\rangle = \langle a^m\rangle$. Then every homomorphism $\xi : H \to \mathbb{R}$ can be extended to a homomorphism $\chi : \Gamma_n \to \mathbb{R}$.

Proof. The equation $\chi|_H = \xi$ is equivalent to a system of r equations

$$\begin{cases} \chi({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}) = \xi({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}), \\ \chi({t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}) = \xi({t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}), \\ \qquad \vdots \\ \chi({t_r}^{k_{rr}}a^{l_r}) = \xi({t_r}^{k_{rr}}a^{l_r}). \end{cases}$$

So, to create such an extension χ we just have to define $\chi(a) = 0$ and define the real numbers $\chi(t_i)$ satisfying equations (1) to (r) above. Equation (r) is equivalent to

$$k_{rr}\chi(t_r) = \xi(t_r^{k_{rr}}a^{l_r}),$$

so if we define $\chi(t_r) = \frac{1}{k_{rr}}\xi(t_r^{k_{rr}}a^{l_r})$, equation (r) is satisfied. Similarly, equation $(r-1)$ is equivalent to

$$k_{r-1,r-1}\chi(t_{r-1}) + k_{r-1,r}\chi(t_r) = \xi(t_{r-1}^{k_{r-1,r-1}}t_r^{k_{r-1,r}}a^{l_{r-1}}),$$

so if we define $\chi(t_{r-1}) = \frac{1}{k_{r-1,r-1}}\xi(t_{r-1}^{k_{r-1,r-1}}t_r^{k_{r-1,r}}a^{l_{r-1}}) - \frac{k_{r-1,r}}{k_{r-1,r-1}}\chi(t_r)$, equation $(r-1)$ is satisfied. By doing this recursively to all i, we are done. □

Theorem 3.8. *Let H be a finite index subgroup of Γ_n, say,*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^m\rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m,n) = 1$ and $H \cap \langle a\rangle = \langle a^m\rangle$, and let $\alpha = a^m$ and $x_i = {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i}$ be its generators. Then $\Sigma^1(H)^c = \{[\xi_1], ..., [\xi_r]\}$, where $\xi_i(x_j) = k_{ji}$ if $j \leq i$ and $\xi_i(x_j) = 0$ if $j > i$.

In other words, if we identify $S(H) \simeq S^{r-1}$ as we did above, then

$$\Sigma^1(H)^c = \left\{ \frac{(k_{11}, 0, 0, ..., 0)}{\|(k_{11}, 0, 0, ..., 0)\|}, \frac{(k_{12}, k_{22}, 0, ..., 0)}{\|(k_{12}, k_{22}, 0, ..., 0)\|}, ..., \frac{(k_{1r}, k_{2r}, k_{3r}, ..., k_{rr})}{\|(k_{1r}, k_{2r}, k_{3r}, ..., k_{rr})\|} \right\}.$$

Proof. By Lemma 3.7, $\Sigma^1(H)^c = i^*(\Sigma^1(\Gamma_n)^c)$ so by Theorem 2.4, $\Sigma^1(H)^c = \{[\chi_1|_H], ..., [\chi_r|_H]\}$. Using that $\chi_i(t_j) = 1$ if $i = j$ and $\chi_i(t_j) = 0$, it is easy to see that the image of $[\chi_i|_H]$ (which we denote by $[\xi_i]$) under the homeomorphism $S(H) \simeq S^{r-1}$ described above is $\frac{(k_{1i},...,k_{ii},0,...,0)}{\|(k_{1i},...,k_{ii},0,...,0)\|}$. This completes the proof. □

3.4. **Finite index subgroups that are not Γ_k.** In [3] it was shown that every finite index subgroup of a solvable Baumslag-Solitar group $BS(1,n)$ is also (isomorphic to) a solvable Baumslag-Solitar group $BS(1,n^k)$ for some $k \geq 1$. Since the groups Γ_n are generalizations of $BS(1,n)$, it is natural to ask whether every finite index subgroup of Γ_n is also (isomorphic to) another Γ_k for some $k \geq 2$. In this section we show that this question has a negative answer. Below, we consider a specific class of finite index subgroups H of Γ_n for which we give necessary and sufficient conditions for H to be isomorphic to Γ_k for some $k \geq 2$.

Theorem 3.9. *Let H be a finite index subgroup of Γ_n such that*

$$H = \langle t_1^{k_{11}}t_2^{k_{12}}...t_r^{k_{1r}}, t_2^{k_{22}}...t_r^{k_{2r}}, ..., t_r^{k_{rr}}, a^m\rangle$$

with $k_{11} > 0$, $0 \leq k_{ij} < k_{ii}$ for all $1 \leq i < j \leq r$ and $m > 0$ such that $\gcd(m,n) = 1$. Then

$$H \simeq \Gamma_k \text{ for some } k \geq 2 \text{ if and only if } k_{ij} = 0 \text{ for all } 1 \leq i < j \leq r.$$

Proof. Suppose first that $k_{ij} = 0$ for all $1 \leq i < j \leq r$. Then from Theorem 3.5 we immediately get that $H \simeq \Gamma_k$ for $k = p_1^{y_1k_{11}}...p_r^{y_rk_{rr}}$. Suppose now that $H \simeq \Gamma_k$ for some $k \geq 2$ and write $k = q_1^{z_1}...q_s^{z_s}$, $q_1 < q_2 < ... < q_s$, $z_i \geq 1$ the prime decomposition of k. Then in particular $s = card(\Sigma^1(\Gamma_k)^c) = card(\Sigma^1(H)^c) = r$, so $k = q_1^{z_1}...q_r^{z_r}$. By Theorem 3.5, H has the presentation

$$H = \langle \alpha, x_1, ..., x_r \mid x_i\alpha x_i^{-1} = \alpha^{n_i},\ x_ix_j = x_jx_i \text{ for all } i,j\rangle,$$

where $n_i = p_i^{y_ik_{ii}}...p_r^{y_rk_{ir}}$. There is also a split exact sequence

$$1 \to ker(\pi) \to H \xrightarrow{\pi} \mathbb{Z}^r \to 1$$

where $\pi(x_i) = e_i$, $\pi(\alpha) = 0$ and $ker(\pi)$ abelian. In particular, every element of H can be written as $x_1^{\lambda_1}...x_r^{\lambda_r}u$ for some $\lambda_i \in \mathbb{Z}$ and $u \in ker(\pi)$. Since $H \simeq \Gamma_k$, then there must be $r+1$ elements inside H (which are the images of the analogous $r+1$ elements in Γ_k), say, $X_i = x_1^{k'_{i1}}...x_r^{k'_{ir}}u_i$, $1 \leq i \leq r$ and $A = x_1^{\tilde{k_1}}...x_r^{\tilde{k_r}}\tilde{u}$ for some $k'_{ij}, \tilde{k_i} \in \mathbb{Z}$ and $u_i, \tilde{u} \in ker(\pi)$, such that $H = \langle X_1, ..., X_r, A\rangle$ and $X_iAX_i^{-1} = A^{q_i^{z_i}}$ for all $1 \leq i \leq r$. By projecting any of these equations on $\mathbb{Z}^r$ we obtain $\tilde{k_1} = ... = \tilde{k_r} = 0$ and so $A = \tilde{u} = x_1^{-\lambda_1}...x_r^{-\lambda_r}\alpha^M x_r^{\lambda_r}...x_1^{\lambda_1}$ for some $\lambda_i \geq 0$ and $M \neq 0$. By replacing this in the r equations above and using that $ker(\pi)$ is abelian and the x_i's commute with each other, we obtain the r equations in H

$$x_1^{k'_{i1}}...x_r^{k'_{ir}}\alpha^M x_r^{-k'_{ir}}...x_1^{-k'_{i1}} = \alpha^{Mq_i^{z_i}} \tag{3.7}$$

for each $1 \leq i \leq r$. If a power k'_{ij} is nonnegative we can use a relation of H to conjugate α^M. If it is negative, though, then since all the x_i commute we can push the two x_j from the left side to the right side of equation (3.7) and use the (now positive) power $-k'_{ij}$ to conjugate $\alpha^{Mq_i^{z_i}}$. Thus equation (3.7) will always imply an equality of a power of α^M with a power of $\alpha^{Mq_i^{z_i}}$. Since H is torsion-free and $M \neq 0$, this yields an equation of prime decomposition which depends on the sign of the k'_{ij}. After a careful analysis of the possible prime decomposition equations we can conclude that k'_{ij} is 1 if $i = j$ and 0 otherwise. The equations (3.7) become $x_i\alpha^M x_i^{-1} = \alpha^{Mp_i^{z_i}}$. This implies $p_i^{y_ik_{ii}}p_{i+1}^{y_{i+1}k_{i,i+1}}...p_r^{y_rk_{ir}} = p_i^{z_i}$, which implies $k_{i,i+1} = ... = k_{ir} = 0$. Since i is arbitrary, we have that $k_{ij} = 0$ for any $1 \leq i < j \leq r$, as desired. □

4. Convex polytopes and property R_∞

In this section we show that finding a special kind of invariant convex polytope in the character sphere $S(G)$ is enough to guarantee property R_∞ for a finitely generated group G (Theorem 4.8). We will use a slightly more general version of Theorem 3.3 in [4], which we state below. The proof is the same given there, just by observing that the authors didn't use directly the definition of $\Sigma^1(G)^c$ but only the fact that it is invariant in $S(G)$ (that is, invariant under all permutations of the form $[\chi] \mapsto [\chi \circ \varphi]$ for $\varphi \in Aut(G)$).

Theorem 4.1. *Let G be a finitely generated group. Suppose there is a nonempty and finite subset $A \subset S(G)$ which is invariant in $S(G)$, consisting only of rational points and contained in an open hemisphere of $S(G)$. Then G has property R_∞.* □

Let G be a finitely generated group whose abelianized group G^{ab} has free rank n. Consider the homeomorphism

$$\begin{aligned} \mathfrak{h} : S(G) &\longrightarrow S^{n-1} \\ [\chi] &\longmapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\|(\chi(x_1), ..., \chi(x_n))\|}, \end{aligned}$$

where the $x_i \in G$ are the free-abelian generators of G^{ab}. Given $\varphi \in Aut(G)$, we have the induced homeomorphism $\varphi^* : S(G) \to S(G)$ with $\varphi^*[\chi] = [\chi \circ \varphi]$. Let $\varphi^S : S^{n-1} \to S^{n-1}$ be the composition $\varphi^S = \mathfrak{h} \circ \varphi^* \circ \mathfrak{h}^{-1}$.

By the definition above, $K \subset S(G)$ is invariant in $S(G)$ if and only if $\mathfrak{h}(K)$ is invariant under φ^S for all $\varphi \in Aut(G)$. From now on, we assume the standard definitions of convex subsets and convex hulls of euclidean spaces $\mathbb{R}^d$. For spherical objects, the definitions will be the following:

Definition 4.2. Let $A \subset S^n \subset \mathbb{R}^{n+1}$ and suppose A is contained in an open hemisphere of S^n, say, $A \subset O(v) = \{x \in S^n \mid \langle x, v\rangle > 0\}$ for some $v \in S^n$. We say that A is (spherically) convex if for any $a_1, a_2 \in A$, $\gamma_{a_1,a_2}(t) = \frac{(1-t)a_1+ta_2}{\|(1-t)a_1+ta_2\|} \in A$ for all $t \in [0, 1]$. The convex hull of any subset $A \subset O(v)$ is the smallest convex subset of $O(v)$ which contains A and is denoted by $conv(A)$.

It is an easy task to show that $conv(A)$ above can be described as

$$conv(A) = \left\{ \frac{t_1a_1 + ... + t_ma_m}{\|t_1a_1 + ... + t_ma_m\|} \mid m \geq 1, a_i \in A, t_i > 0 \right\}.$$

The following lemma shows a special property of the homeomorphisms φ^S.

Lemma 4.3. *The homeomorphism $\varphi^S : S^{n-1} \to S^{n-1}$ maps convex hulls to convex hulls. Precisely, let $A \subset O(v)$ and suppose $\varphi^S(A) \subset O(w)$ for some w. Then $\varphi^S(conv(A)) = conv(\varphi^S(A))$.*

Proof. Since $(\varphi^{-1})^S = (\varphi^S)^{-1}$, it is enough to show that $\varphi^S(conv(A)) \subset conv(\varphi^S(A))$. Let $P \in conv(A)$ and write $P = \frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$ for some $a_i \in A$ and $t_i > 0$. For each a_i, since $\mathfrak{h} : S(G) \to S^{n-1}$ is surjective we write $a_i = \mathfrak{h}[\chi_i]$ and by multiplying the representative χ_i by some $r > 0$ if necessary we can actually suppose $a_i = \mathfrak{h}[\chi_i] = (\chi_i(x_1), ..., \chi_i(x_n))$. Then, by definition, $\varphi^S(a_i) = \frac{1}{\lambda_i}(\chi_i \circ \varphi(x_1), ..., \chi_i \circ \varphi(x_n))$, where $\lambda_i = \|(\chi_i \circ \varphi(x_1), ..., \chi_i \circ \varphi(x_n))\| > 0$. Now we compute $\varphi^S(P)$. It is easy to see that $\mathfrak{h}[t_1\chi_1 + ... + t_m\chi_m] = P$, since $a_i = \mathfrak{h}[\chi_i]$. By denoting

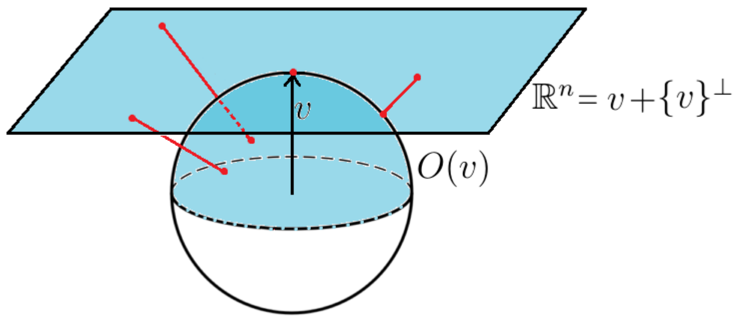
$$\lambda = \|(t_1(\chi_1 \circ \varphi)(x_1) + ... + t_m(\chi_m \circ \varphi)(x_1), ..., t_1(\chi_1 \circ \varphi)(x_n) + ... + t_m(\chi_m \circ \varphi)(x_n))\|,$$

we have

$$\begin{aligned}
\varphi^S(P) &= \frac{t_1}{\lambda}((\chi_1 \circ \varphi)(x_1), ..., (\chi_1 \circ \varphi)(x_n)) + ... + \frac{t_m}{\lambda}((\chi_m \circ \varphi)(x_1), ..., (\chi_m \circ \varphi)(x_n)) \\
&= \frac{\lambda_1 t_1}{\lambda}\varphi^S(a_1) + ... + \frac{\lambda_m t_m}{\lambda}\varphi^S(a_m) \\
&= \frac{\frac{\lambda_1 t_1}{\lambda}\varphi^S(a_1) + ... + \frac{\lambda_m t_m}{\lambda}\varphi^S(a_m)}{\|\frac{\lambda_1 t_1}{\lambda}\varphi^S(a_1) + ... + \frac{\lambda_m t_m}{\lambda}\varphi^S(a_m)\|} \quad \text{(since the above vector is already unitary)} \\
&\in conv(\varphi^S(A)),
\end{aligned}$$

as desired. □

Given an open hemisphere $O(v) = \{x \in S^n \mid \langle x, v\rangle > 0\}$ of S^n for some $v \in S^n$, consider the affine n-space $v + \{v\}^\perp = \{v + w \mid \langle w, v\rangle = 0\} \subset \mathbb{R}^{n+1}$. One can show that there is a homeomorphism $\theta_v : v + \{v\}^\perp \to O(v)$ with $\theta_v(P) = \frac{P}{\|P\|}$, the inverse map given by $P \mapsto \frac{\|v\|^2}{\langle P,v\rangle} P$ (see next figure). From now on we identify $\mathbb{R}^n = v + \{v\}^\perp$.



It is straightforward to show that $\theta_v : \mathbb{R}^n \to O(v)$ maps convex hulls of $\mathbb{R}^n$ to convex hulls of $O(v)$. Now we will define the convex polytopes in our context.

Definition 4.4 (Euclidean convex polytopes)**.** A closed halfspace in $\mathbb{R}^d$ is a set of the form $H = \{x \in \mathbb{R}^d \mid \langle x, v\rangle \geq \beta\}$ for some $0 \neq v \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$. A convex polytope K in $\mathbb{R}^d$ is a finite intersection $K = \cap_{i=1}^n H_i$ of closed halfspaces H_i which is also a bounded subset. Thinking of K as a submanifold of $\mathbb{R}^d$ (with boundary), there is a well defined dimension $r = dim(K)$, so we say that K is an r-polytope.

We can always suppose that the family $\{H_i\}$ of closed halfspaces defining K is irredundant, that is, is the minimal family necessary to define K.

Definition 4.5 (Spherical convex polytopes)**.** For any $n \geq 0$, a closed hemisphere in S^n is a set having the form $C(w) = \{p \in S^n \mid \langle p, w\rangle \geq 0\}$ for some $w \in S^n$. A convex polytope $K \subset S^n$ is a finite intersection of closed hemispheres in S^n. Given a finitely generated group G with $S(G) \overset{\mathfrak{h}}{\simeq} S^{n-1}$, we say that $K \subset S(G)$ is a convex polytope if $\mathfrak{h}(K)$ is a convex polytope in S^{n-1}.

The next lemma uses some known facts about Euclidean polytopes with which we will assume the reader is familiar.

Lemma 4.6. *Let $K \subset \mathbb{R}^d$ be a (Euclidean) d-polytope (maximal dimension) and $f : K \to K$ a homeomorphism. If f maps segments to segments, that is, for any $P, Q \in K$, $f(conv(P,Q)) = conv(f(P), f(Q))$, then f maps vertices to vertices.*

Proof. Let $K = \cap_{i=1}^n H_i$ for an irredundant family $\{H_i\}$ and let $F_i = K \cap H_i$ be its facets. It is known that $n \geq d+1$, that $\partial K = F_1 \cup ... \cup F_n$ and that a point of K is a vertex if and only if it belongs to at least d different facets. Since f is a homeomorphism, it must map the boundary ∂K to itself, and so $f(F_1 \cup ... \cup F_n) = F_1 \cup ... \cup F_n$. Suppose by contradiction that a vertex $P \in K$ is mapped to a non-vertex point $f(P) \in K$ (but obviously $P, f(P) \in \partial K$). If a point $Q \in K$ belongs to any facet of K containing P (say, F), then $conv(Q, P) \subset F$, since every facet is convex. Then $conv(f(Q), f(P)) \subset f(F) \subset \partial K$ by hypothesis, so the whole straight path joining $f(Q)$ and $f(P)$ is contained in the boundary ∂K. Then one can show that $f(Q)$ must be in a facet which also contains $f(P)$. This argument shows that all the facets containing P must be mapped into the facets containing $f(P)$. But there are at least d facets containing P, say, $F_1, ..., F_d$ and at most $d-1$ facets containing $f(P)$, say, $F_{i_1}, ..., F_{i_{d-1}}$. Then

$$f(F_1 \cup ... \cup F_d) \subset F_{i_1} \cup ... \cup F_{i_{d-1}}.$$

We continue: since there are at least $d+1$ facets, let $Z \in \partial K$ be a point outside $F_{i_1} \cup ... \cup F_{i_{d-1}}$, say, $Z \in F_{i_d}$, and we can suppose F_{i_d} is the only facet containing Z. Since f is surjective, $Z = f(W)$, so W must be a boundary point outside $F_1 \cup ... \cup F_d$, say, $W \in F_{d+1}$. By the same argument above, we must have $f(F_{d+1}) \subset F_{i_d}$ and so $f(F_1 \cup ... \cup F_{d+1}) \subset F_{i_1} \cup ... \cup F_{i_d}$. If $d+1 = n$, we stop. If not, we follow these same steps. After a finite number of steps we will have

$$f(F_1 \cup ... \cup F_n) \subset F_{i_1} \cup ... \cup F_{i_{n-1}},$$

so $f(\partial K) \subsetneq \partial K$, contradiction. □

Theorem 4.7. *Let G be a finitely generated group and $K \subset S(G)$ a convex polytope contained in an open hemisphere of $S(G)$. Then K is invariant in $S(G)$ if and only if $V(K)$ is invariant in $S(G)$.*

Proof. The convex polytope $\mathfrak{h}(K)$ is contained in some open hemisphere $O(v)$ of S^{n-1}. Let $\theta_v : \mathbb{R}^{n-1} \to O(v)$ be the homeomorphism previously defined. One can verify from the definition of θ_v that the preimage of a closed hemisphere in S^{n-1} under θ_v is a closed halfspace in $\mathbb{R}^{n-1}$. Then to see that the preimage $K' = {\theta_v}^{-1}(\mathfrak{h}(K))$ is a polytope it suffices to see that it is bounded. Since $\mathfrak{h}(K)$ is closed in the compact S^{n-1}, it is compact. Since θ_v is a homeomorphism, K' is also compact in $\mathbb{R}^{n-1}$ and therefore bounded, so it is in fact a r-polytope for some $0 \leq r \leq n-1$.

To show the theorem, let $\varphi \in Aut(G)$. It is enough to show that $\mathfrak{h}(K)$ is invariant under φ^S if and only if $V(\mathfrak{h}(K))$ is. Suppose first that $V(\mathfrak{h}(K))$ is invariant under φ^S. In Euclidean space, every convex polytope is the convex hull of its vertices. Since θ_v maps convex hulls to spherical convex hulls, it follows that $\mathfrak{h}(K)$ is also the convex hull of its vertices. Using Lemma 4.3, we have

$$\varphi^S(\mathfrak{h}(K)) = \varphi^S(conv(V(\mathfrak{h}(K)))) = conv(\varphi^S(V(\mathfrak{h}(K)))) = conv(V(\mathfrak{h}(K))) = \mathfrak{h}(K),$$

as desired. Now, suppose $\varphi^S(\mathfrak{h}(K)) = \mathfrak{h}(K)$. If $r < n-1$, then K' is contained in a proper r-hyperspace of $\mathbb{R}^{n-1}$, say, E^r. There is a linear isomorphism and isometry $T : \mathbb{R}^r \to E^r$ and a r-polytope $\tilde{K} \subset \mathbb{R}^r$ such that $K' = T(\tilde{K})$. Consider the composition of homeomorphisms

$$\tilde{K} \xrightarrow{T} K' \xrightarrow{\theta_v} \mathfrak{h}(K) \xrightarrow{\varphi^S} \mathfrak{h}(K) \xrightarrow{{\theta_v}^{-1}} K' \xrightarrow{T^{-1}} \tilde{K}.$$

Since T maps straight paths to straight paths, θ_v maps straight paths to geodesic paths and φ^S maps geodesic paths to geodesic paths, this composition is a homeomorphism which maps straight paths to straight paths. Since $\tilde{K}$ has maximal dimension in $\mathbb{R}^r$, by Lemma 4.6 this composition must map the vertices of $\tilde{K}$ to themselves. Since the vertices of $\mathfrak{h}(K)$ are the image of the ones from K', it follows that φ^S must map the vertices of $\mathfrak{h}(K)$ to themselves, as desired. If K' already had maximal dimension $r = n-1$, the proof is the same, but we don't even need to use $\tilde{K}$ and T. □

Theorem 4.8. *Let G be a finitely generated group. If there is a convex polytope $K \subset S(G)$ contained in an open hemisphere of $S(G)$ and with rational vertices such that it is invariant under all homeomorphisms induced by automorphisms of G, then G has property R_∞. In particular, if $\Sigma^1(G)^c$ is one such polytope, then G has property R_∞.*

Proof. By the previous theorem, $V(K) \subset S(G)$ is finite, invariant and by definition contained in an open half-space of $S(G)$. Then the result follows directly from Theorem 4.1. □

5. Property R_∞ for Γ_n, its finite index subgroups, and direct products

In this section we use all the information previously gathered to guarantee property R_∞ for Γ_n (Corollary 5.2), its finite index subgroups H (Corollary 5.3) and also for any (finite) direct product involving these groups (Corollary 5.4). Note that property R_∞ is already known for Γ_n and its finite index subgroups (see [11]). However, by using sigma theory, we obtain the same results with new and easier proofs. Corollary 5.4 for the direct product was not considered in [11]. In Proposition 5.6, we exhibit a group G where Theorem 4.8 can be used to guarantee property R_∞ without the need of completely computing the Σ^1 invariant.

We will make use of the following theorem.

Theorem 5.1 ([4], Theorem 3.3)**.** *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\}$$

is a (nonempty) finite set of rational points. If $\{[\chi_1], ..., [\chi_m]\}$ is contained in an open hemisphere of $S(G)$, then G has property R_∞.

Corollary 5.2. *The generalized solvable Baumslag-Solitar groups Γ_n have property R_∞.*

Proof. Observe that, by Theorem 2.4, $\Sigma^1(\Gamma_n)^c$ is a finite set of rational points and is contained in the open hemisphere $O\left(\frac{(1,1,...,1)}{\|(1,1,...,1)\|}\right)$. The result follows from Theorem 5.1. □

Corollary 5.3. *All finite index subgroups of Γ_n have property R_∞.*

Proof. Let H be such finite index subgroup. As above, just observe that, by Theorem 3.8, $\Sigma^1(H)^c$ is a finite set of rational points and is contained in the open hemisphere $O\left(\frac{(1,1,...,1)}{\|(1,1,...,1)\|}\right)$ of $S(H)$. The result follows from Theorem 5.1. □

Now we show property R_∞ for any (finite) direct product between the groups Γ_n and its finite index subgroups.

Corollary 5.4. *Let $G = G_1 \times ... \times G_m$, where each G_i is some Γ_n or some finite index subgroup H of Γ_n. Then G has R_∞ property.*

Proof. By Theorems 2.4, 3.8 and by the known formula for the Σ^1 invariant of a direct product of groups (Proposition $A2.7$ of [9], for example), we easily see that $\Sigma^1(G)^c$ is a finite set of rational points of $S(G)$. Furthermore, by Theorems 2.4 and 3.8, we know that $\Sigma^1(G_i)^c$ is contained in an open hemisphere $O(v_i)$ of $S(G_i)$, for every i. From that, it is easy to see that $\Sigma^1(G)^c$ is contained in the open hemisphere $O(v_1, ..., v_m)$ of $S(G)$. The result follows from Theorem 5.1. □

Let G be a finitely generated group and X a finite set of generators for G. A path in the Cayley graph $\Gamma = \Gamma(G, X)$ of G is denoted by $p = (g, y_1 ... y_n)$. The path p starts at g, walks through the edge (g, y_1) until the vertex gy_1, walks through (gy_1, y_2) until gy_1y_2 and so on, until its terminus $gy_1...y_n$. Given $\chi \in Hom(G, \mathbb{R})$, the evaluation function ν_χ is given by

$$\nu_\chi(p) = \min\{\chi(g), \chi(gy_1), ..., \chi(gy_1...y_n)\}.$$

We are going to use the following geometric Σ^1-criterion given by R. Strebel (Theorem $A3.1$) in [9] in Proposition 5.6 to illustrate a situation where we can use Theorem 4.8 to guarantee property R_∞ for a finitely generated group G without having to completely compute $\Sigma^1(G)$.

Theorem 5.5 (Geometric Criterion for Σ^1)**.** *Let G be a finitely generated group with finite generating set X and denote $Y = X^{\pm}$. Let $[\chi] \in S(G)$ and choose $t \in Y$ such that $\chi(t) > 0$. Then the following are equivalent:*

1) *Γ_χ is connected (or $[\chi] \in \Sigma^1(G)$);*
2) *For every $y \in Y$, there exists a path p_y from t to yt in Γ such that $\nu_\chi(p_y) > \nu_\chi((1, y))$.*

Proposition 5.6. *Let*

$$G = \langle a, t, s \mid tat^{-1} = a^n,\ sas^{-1} = a^m,\ tst^{-1}s^{-1} = a^r\rangle$$

for some coprime numbers $n, m \geq 2$ and some $r \in \mathbb{Z}$. Then G has property R_∞.

Proof. We have the homeomorphism $\mathfrak{h} : S(G) \to S^1$, sending $[\chi]$ to the normalized of $(\chi(t), \chi(s))$. Let us compute $\Sigma^1(G)$ by the geometric criterion. Fix $X = \{a, t, s\}$ and $Y = \{a, a^{-1}, t, t^{-1}, s, s^{-1}\}$.

1) if $\chi(t) < 0$ then $[\chi] \in \Sigma^1(G)$. Fix t^{-1} such that $\chi(t^{-1}) > 0$. By using the relations on G, one can see that the paths $p_a = (t^{-1}, a^n)$, $p_{a^{-1}} = (t^{-1}, a^{-n})$, $p_t = (t^{-1}, t)$, $p_{t^{-1}} = (t^{-1}, t^{-1})$, $p_s = (t^{-1}, a^r s)$ and $p_{s^{-1}} = (t^{-1}, s^{-1}a^{-r})$ satisfy 2) of 5.5, so $[\chi] \in \Sigma^1(G)$.
2) if $\chi(s) < 0$ then $[\chi] \in \Sigma^1(G)$. Similar to item 1).
3) if $\chi(t) = 1$ and $\chi(s) = 0$ then $[\chi] \notin \Sigma^1(G)$.

   Suppose by contradiction that $[\chi] \in \Sigma^1(G)$. Then, in particular, there is a path $p = (1, w)$ in Γ_χ from 1 to $t^{-1}at$. Write

   $$w = t^{k_{11}} s^{k_{12}} a^{r_1} ... t^{k_{c1}} s^{k_{c2}} a^{r_c}.$$

   Since p is contained in Γ_χ, $\chi(t) = 1$ and $\chi(s) = 0$ we must have

   $$k_{11} \geq 0,\ k_{11} + k_{21} \geq 0,\ ...,\ k_{11} + ... + k_{c-1,1} \geq 0 \text{ and } k_{11} + ... + k_{c1} = 0.$$

   By using the relations on G, we push right $t^{k_{11}}$ until $t^{k_{21}}$, then we push right $t^{k_{11}+k_{21}}$ until $t^{k_{31}}$, and so on. Since $k_{11} + ... + k_{c1} = 0$, we eliminate from w all the t-letters and (after relabeling the s and a powers) we can write $w = s^{k_1} a^{r_1} ... s^{k_c} a^{r_c}$ in G. But, as a vertex, w must be the end of the path p. So we have $w = t^{-1}at$ and therefore

   $$a = twt^{-1} = t(s^{k_1} a^{r_1} ... s^{k_c} a^{r_c})t^{-1} = (a^r s)^{k_1} a^{nr_1} ... (a^r s)^{k_c} a^{nr_c},$$

   or

   $$w' = (a^r s)^{k_1} a^{nr_1} ... (a^r s)^{k_{c-1}} a^{nr_{c-1}} (a^r s)^{k_c} a^{nr_c - 1} = 1$$

in G. By projecting this equation onto the s-coordinate, we have $k_1 + ... + k_c = 0$. Also, $(a^r s)a^M = a^{mM}(a^r s)$ and $a^M(a^r s)^{-1} = (a^r s)^{-1}a^{mM}$ for every $M \in \mathbb{Z}$. This means that, in w', the entire positive pieces $(a^r s)^{k_i}$ can be pushed right and the negative ones can be pushed left. After doing this, we obtain an expression of the form

$$(a^r s)^{-\lambda} a^{\alpha_1 nr_1 + ... + \alpha_{c-1} nr_{c-1} + \alpha_c(nr_c - 1)} (a^r s)^{\lambda} = 1,$$

where each α_i is either 1 or a positive power of m. This easily implies

$$\alpha_1 nr_1 + ... + \alpha_{c-1} nr_{c-1} + \alpha_c(nr_c - 1) = 0.$$

By putting all the multiples of n above to the left and only α_c on the right, we get either $Mn = 1$ (contradiction with the fact $n \geq 2$) or $Mn = m^Q$ for $Q \geq 1$ (contradiction with the fact $\gcd(n, m) = 1$). This shows item 3).

4) if $\chi(t) = 0$ and $\chi(s) = 1$ then $[\chi] \notin \Sigma^1(G)$. Similar to item 3).

Now identify $S(G)$ with S^1 by the homeomorphism $\mathfrak{h}$ and let $[\chi_1]$ and $[\chi_2]$ be the points of items 3) and 4), respectively. Items 1) and 2) showed that the geodesic γ in $S(G)$ between these points contains $\Sigma^1(G)^c$. We claim that γ is invariant in $S(G)$. In fact, if $\varphi \in Aut(G)$ and $p \in \gamma$, then by Lemma 4.3 $\varphi^*(p)$ must be in the geodesic between $\varphi^*[\chi_1]$ and $\varphi^*[\chi_2]$. By the Σ invariance and by items 3) and 4), $\varphi^*[\chi_1]$ and $\varphi^*[\chi_2]$ are in $\Sigma^1(G)^c$; therefore, by items 1) and 2), they must be in γ. Since γ is a convex subset we have $\varphi^*(p) \in \gamma$, which shows our claim. Thus, in $S(G)$ we have γ an invariant convex 1-dimensional polytope with the two rational vertices $[\chi_i]$ and the proposition follows from Theorem 4.8. □

Remark 5.7. In Proposition 5.6, if $r \neq 0$, we do not know whether the group G is metabelian in general. While in such cases the proof of Theorem 2.4 does not necessarily apply, the geometric criterion does apply. Of course, if $r = 0$, we have $G = \Gamma(S)$ for $S = \{n, m\}$, so G is metabelian. Therefore, Proposition 5.6 illustrates an alternative way to derive property R_∞ besides using the BNS invariant Σ^1.

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