

Bates College

**SCARAB**

---

All Faculty Scholarship

Departments and Programs

---

4-15-2021

## Twisted conjugacy in fundamental groups of geometric 3-manifolds

Daciberg Gonçalves  
*Universidade de São Paulo*

Parameswaran Sankaran  
*Chennai Mathematical Institute*

Peter Wong  
*Bates College, pwong@bates.edu*

Follow this and additional works at: [https://scarab.bates.edu/faculty\\_publications](https://scarab.bates.edu/faculty_publications)

---

### Recommended Citation

Goncalves, D, Sankaran, P. and Wong, P. 2021. "Twisted conjugacy in fundamental groups of geometric 3-manifolds." 293. <https://doi.org/10.1016/j.topol.2020.107568>

This Article is brought to you for free and open access by the Departments and Programs at SCARAB. It has been accepted for inclusion in All Faculty Scholarship by an authorized administrator of SCARAB. For more information, please contact [batesscarab@bates.edu](mailto:batesscarab@bates.edu).

# TWISTED CONJUGACY IN FUNDAMENTAL GROUPS OF GEOMETRIC 3-MANIFOLDS

DACIBERG GONÇALVES, PARAMESWARAN SANKARAN, AND PETER WONG

ABSTRACT. A group  $G$  has the  $R_\infty$ -property if for every  $\varphi \in \text{Aut}(G)$ , there are an infinite number of  $\varphi$ -twisted conjugacy classes of elements in  $G$ . In this note, we determine the  $R_\infty$ -property for  $G = \pi_1(M)$  for all geometric 3-manifolds  $M$ .

## 1. INTRODUCTION

Let  $M$  be a closed connected  $n$ -manifold and  $f : M \rightarrow M$  a selfmap. Classical Nielsen fixed point theory is concerned with the minimal number of fixed points among all maps homotopic to  $f$ , i.e., the number  $MF[f] := \min_{g \sim f} \{\#(\text{Fix}g = \{x \in M \mid g(x) = x\})\}$ . If  $n \geq 3$ , a classical theorem of Wecken asserts that  $MF[f] = N(f)$ , the Nielsen number of  $f$ . For  $n = 2$ , the difference  $MF[f] - N(f)$  can be arbitrarily large. For  $n \geq 3$ , the computation of  $N(f)$  is a central issue but is very difficult in general. When  $M$  is a Jiang-type space, for instance a generalized lens space, an orientable coset space of a compact connected Lie group, a spherical space form, or a nilmanifold, either  $N(f) = 0$  or  $N(f) = R(f)$ , the Reidemeister number of  $f$ . If  $\varphi$  is the induced homomorphism of  $f$  on  $\pi_1(M)$ ,  $R(f) = R(\varphi)$ , the cardinality of the set of  $\varphi$ -twisted conjugacy classes of elements in  $\pi_1(M)$ . In such a situation, if  $R(f) = \infty$  then  $N(f) = 0$  which implies that  $f$  is homotopic to a fixed point free map. For example, for any  $n \geq 5$ , there exists an  $n$ -dimensional nilmanifold  $M$  such that every homeomorphism  $f : M \rightarrow M$  is isotopic to a fixed point free map [GW2]. This result is a consequence of the  $R_\infty$ -property of  $\pi_1(M)$  for certain nilmanifolds  $M$ . Recall that a group  $G$  has the  $R_\infty$ -property if for every  $\varphi \in \text{Aut}(G)$ , the set of orbits of the left action  $\sigma \cdot \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}$  is infinite. It is therefore natural to ask for what families of  $n$ -manifolds  $M$  does  $\pi_1(M)$  have the  $R_\infty$ -property. For  $n = 3$ , the Thurston-Perelman Geometrization Theorem asserts that every closed 3-manifold is made up of finite pieces of 3-manifolds equipped with geometries of the following eight types: (I)  $S^3$  (Spherical); (II)  $S^2 \times \mathbb{R}$ ; (III)  $\mathbb{E}^3$  (Euclidean); (IV) Nil; (V)  $SL(2, \mathbb{R})$ ; (VI)  $\mathbb{H}^2 \times \mathbb{R}$ ; (VII) Sol; (VIII)  $\mathbb{H}^3$  (Hyperbolic). By a *geometric* 3-manifold, we mean a connected 3-manifold equipped with a geometry from (I) - (VIII) with finite

---

*Date:* June 2, 2020.

*2010 Mathematics Subject Classification.* 20E45, 20E36, 57M20, 55M20.

*Key words and phrases.* Twisted conjugacy, geometric three-manifolds, Nielsen fixed point theory.

The first author is partially supported by Projeto Temático-FAPESP Topologia Algébrica, Geometria e Diferencial 2016/24707-4 (São Paulo-Brazil). The first and third authors thank the IMSc (August 2018) and the CMI, Chennai (December 2019), for their support during their visits. Second and third authors thank the IME-USP, São Paulo for its support during the authors' visit in February 2019.

volume (see [Wi]). It turns out that a geometric 3-manifold is compact except in case of the following geometries where the manifold can be either compact or non-compact:  $\widetilde{SL(2, \mathbb{R})}, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^3$ .

The main objective of this note is to determine whether the fundamental group of a geometric 3-manifold has the  $R_\infty$ -property. Leaving out the case of spherical geometry where the fundamental group is finite, our main result is the following:

**Main Theorem.** *Let  $M$  be a geometric 3-manifold with infinite fundamental group. Then  $\pi_1(M)$  has the  $R_\infty$ -property when  $M$  has any of the following geometries:  $\widetilde{SL(2, \mathbb{R})}, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^3$ . In the remaining cases,  $\pi_1(M)$  has the  $R_\infty$ -property with the following exceptions:*

- (a)  $S^2 \times S^1$ -geometry:  $M \cong S^2 \times S^1, S^2 \tilde{\times} S^1, \mathbb{R}P^2 \times S^1$ ,
- (b)  $\mathbb{E}^3$ -geometry: *The orientable manifolds with holonomy group  $\{1\}, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$ ,*
- (c) *Nil-geometry: The circle bundles over the torus  $S^1 \times S^1$  with non-zero Euler class  $k$ , and, Seifert fibre spaces with base the sphere  $S^2$  having four singular points of type  $(2, 1)$  and holonomy group  $\mathbb{Z}_2$ .*
- (d) *Sol-geometry: The manifolds having this geometry are of two kinds:  $M_0 = T \times I / \sim$ , where the boundary tori are glued via a Anosov diffeomorphism;  $M_1 = E_0 \sqcup E_1 / \sim$ , where  $E_0, E_1$  are twisted  $I$ -bundles over the Klein bottle and their boundary tori are glued via an Anosov diffeomorphism.*

(1) *The group  $\pi_1(M_0) = G = \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}$  where the  $\mathbb{Z}$ -action  $\theta$  on  $\mathbb{Z}^2$  is given by an Anosov matrix  $A \in SL(2, \mathbb{Z})$ . Then  $G$  has the  $R_\infty$ -property if and only if any of the following holds: (i)  $\det(A) = -1$ , (ii)  $A, A^{-1} \in SL(2, \mathbb{Z})$  are not conjugates in  $GL(2, \mathbb{Z})$ , (iii)  $A, A^{-1}$  are conjugates in  $GL(2, \mathbb{Z})$  but are not conjugate to a matrix of the form  $\begin{pmatrix} r & s \\ s & u \end{pmatrix}$ , and, furthermore, neither  $A$  nor  $-A$  equals  $X^p$  for some  $p \in \mathbb{Z}$  and  $X \in GL(2, \mathbb{Z})$  with  $\det(X) = -1$ .*

(2) *The group  $\pi_1(M_1)$  has the  $R_\infty$ -property.*

We have a related notion of a manifold possessing the  $R_\infty$ -property.

**Definition 1.** *We say that a manifold  $M$  has the  $R_\infty$ -property if, for every self-homotopy equivalence  $f : M \rightarrow M$ , the Reidemeister number of the automorphism  $f_\# : \pi_1(M) \rightarrow \pi_1(M)$  is infinite.*

Note that when  $M$  is an aspherical space  $K(\pi, 1)$ , the topological and the algebraic notions of the  $R_\infty$ -properties coincide, that is, an aspherical manifold  $M$  has the  $R_\infty$ -property if and only if the group  $\pi_1(M) = \pi$  has the  $R_\infty$ -property. Other than spherical- and  $S^2 \times \mathbb{R}$ -geometries, the universal covers of the remaining geometric 3-manifolds are diffeomorphic to  $\mathbb{R}^3$  and so they are aspherical. Since the fundamental groups of manifolds admitting spherical geometry are finite, the two notions trivially coincide. We will show that in the remaining case of  $S^2 \times \mathbb{R}$ -geometry also, the two notions agree. In general, however, examples of smooth compact manifolds are known which have the  $R_\infty$ -property but their fundamental groups do not. (See the Appendix.) We remark here that, in the

case of 3-manifolds, the Borel conjecture is known to be valid: if  $M, M'$  are closed aspherical 3-manifolds, any isomorphism  $\phi : \pi_1(M) \rightarrow \pi_1(M')$  is induced by a *homeomorphism*  $f : M \rightarrow M'$  (see [AFW, §2.1]).

The proof of the Main Theorem will be spread over several sections, depending on the type of the geometry under consideration. In many cases, the proof can be found or can be derived from results available in the literature. But they are scattered in various papers and often do not specifically address the case of fundamental groups of geometric 3-manifolds. Specifically, the case of hyperbolic geometry follows from the work of Levitt and Lustig (for compact manifolds) and that of Fel'shtyn (for non-compact ones). In the cases of  $\mathbb{E}^3$ -,  $S^2 \times S^1$ -, and *Sol*-geometries, the proof (for the most part) follows from the work of Gonçalves, Wong, and Zhao. The result for  $\mathbb{E}^3$ -geometry was also obtained by Dekimpe and Penninckx, who considered the more general case of three-dimensional crystallographic groups. The complete result in the case of Nil-geometry is due to Dekimpe [DP, Theorem 4.4]. However, the results on  $\mathbb{H}^2 \times \mathbb{R}$ - and  $SL(2, \mathbb{R})$ -geometries and the *complete* classification of manifolds admitting *Sol*-geometry whose fundamental groups do *not* have the  $R_\infty$ -property, could not be found in the literature.

Our aim here is to present a coherent discussion of all the eight geometries, considering the importance of the role of the fundamental group in the study of 3-manifolds. For the  $R_\infty$ -property of fundamental groups of non-prime 3-manifolds see [GSW].

For the rest of the paper, we leave out the case of spherical geometry.

## 2. GEOMETRIES $\mathbb{H}^3$ AND $\mathbb{E}^3$

**2.1.  $\mathbb{H}^3$ , the hyperbolic geometry.** The fundamental group of a compact hyperbolic 3-manifold is known to be a (torsion-free) non-elementary word hyperbolic group. It follows from the main result of [LL] that these groups all have the  $R_\infty$ -property. The fundamental group of a non-compact, finite volume hyperbolic 3-manifold is relatively hyperbolic (with respect to the finite collection of fundamental groups at the cusps). In this case, Fel'shtyn [F] has shown that such a group has the  $R_\infty$ -property (see also [MS1] and [MS2]).

**2.2.  $\mathbb{E}^3$ , Euclidean or flat geometry.** In this case the  $R_\infty$ -property has been studied in a more general context. Here we shall confine ourselves to the case of 3-manifolds.

Any such manifold  $M$  is a quotient  $\mathbb{R}^3/\pi$  where  $\pi$  is a torsion-free lattice in the group  $Iso(\mathbb{R}^3)$  of isometries of  $\mathbb{R}^3$  and is finitely covered by the 3-torus  $\mathbb{R}^3/\mathbb{Z}^3$ . Thus  $M$  is compact and the fundamental group  $\pi$  of  $M$  therefore admits a finite index subgroup isomorphic to  $\mathbb{Z}^3$ . It turns out that  $\pi$  has a *unique* maximal normal abelian group  $\Gamma \cong \mathbb{Z}^3$ . In particular,  $\Gamma$  is characteristic in  $\pi$ . It is the translation part of  $\pi$ . The group  $\Phi := \pi/\Gamma$ , which is finite, is the *holonomy group* of  $M$ .  $\Phi$  acts on  $\Gamma$  as automorphisms. This is the same as the action of the deck transformation group of the covering  $\mathbb{R}^3/\Gamma \rightarrow M$ . Thus we have an exact sequence in which  $\Gamma \cong \mathbb{Z}^3$  is characteristic and  $\Phi$ , finite:

$$(1) \quad 1 \rightarrow \Gamma \rightarrow \pi \rightarrow \Phi \rightarrow 1.$$

It is known that the fixed subgroup  $\Gamma^\Phi$  equals the centre  $Z(\pi)$  of  $\pi$ . When  $Z(\pi) \cong \mathbb{Z}$ , the quotient  $\pi/Z(\pi)$  is a planar crystallographic group  $\Lambda$ . Irrespective of the rank of  $Z(\Gamma)$ , one has a projection of  $\pi$  onto a planar crystallographic group  $\Lambda$ . Thus one has an exact sequence

$$(2) \quad 1 \rightarrow Z \rightarrow \pi \rightarrow \Lambda \rightarrow 1.$$

However, only the case when  $Z = Z(\pi) \cong \mathbb{Z}$  will be relevant for our purposes.

When  $M$  is non-orientable, it turns out that  $M$  fibres over a circle with fibre the Klein bottle. This results in an exact sequence

$$(3) \quad 1 \rightarrow \pi_1(K) \rightarrow \pi \rightarrow \mathbb{Z} \rightarrow 1.$$

Using the notation of [GWZ1], up to diffeomorphism, there are a total of ten flat 3-manifolds whose fundamental groups  $\pi$  are listed below, where the first six are orientable and the remaining four are non-orientable. We also indicate the holonomy group  $\Phi$  and the centre  $Z(\pi)$ . Whenever it is relevant for our purposes, we shall indicate the planar crystallographic group  $\Lambda = \pi/Z(\pi)$  with  $Z(\pi) \cong \mathbb{Z}$ . We will use the notation of Lyndon [L] for planar crystallographic groups.

We denote the image of an element  $\gamma \in \pi$  under the projection  $\pi \rightarrow \Phi$  by  $\bar{\gamma}$ .

1.  $\langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \{1\}$ .
2.  $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^2, t\alpha_2 t^{-1} = \alpha_2^{-1}, t\alpha_3 t^{-1} = \alpha_3^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_2$ .  $Z(\pi) = \langle \alpha_1 \rangle$ .  $\Lambda = \pi/Z(\pi) \cong G_2$ .
3.  $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^3, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1} \alpha_3^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_3$ .  $Z(\pi) = \langle \alpha_1 \rangle$ .  $\Lambda = \pi/Z(\pi) \cong G_3$ .
4.  $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^4, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_4$ .  $Z(\pi) = \langle \alpha_1 \rangle$ .  $\Lambda = \pi/Z(\pi) \cong G_4$ .  $\Lambda = \pi/Z(\pi) \cong G_4$ .
5.  $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^6, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1} \alpha_3, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_6$ .  $Z(\pi) = \langle \alpha_1 \rangle$ .  $\Lambda = \pi/Z(\pi) \cong G_6$ .
6.  $\langle \alpha_1, \alpha_2, \alpha_3, t_1, t_2, t_3 \mid \alpha_1 \alpha_3 = t_3 t_2 t_1, \alpha_i = t_i^2, t_i \alpha_j t_i^{-1} = \alpha_j^{-1}$  for  $i \neq j, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$  with holonomy  $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ , generated by  $\bar{t}_1, \bar{t}_2$ .  $Z(\pi) = \{1\}$ .
7.  $\pi_1(K) \times \mathbb{Z} = \langle \alpha, \beta \mid \beta \alpha \beta^{-1} = \alpha^{-1} \rangle \times \langle t \rangle$  where  $K$  is the Klein bottle, with holonomy  $\Phi = \mathbb{Z}_2$ , generated by  $\bar{\beta}$ .  $Z(\pi) = gp\langle \beta^2, t \rangle \cong \mathbb{Z}^2$ .
8.  $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \alpha \beta \rangle$  with holonomy  $\Phi = \mathbb{Z}_2$  generated by  $\bar{\beta}$ .  $Z(\pi) = \langle \beta^2 \rangle \cong \mathbb{Z}$ . We have  $gp\langle \alpha, \beta \rangle \cong \pi_1(K)$  and so  $\pi = \pi_1(K) \times \langle t \rangle$ .
9.  $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \beta^{-1} \rangle$  with holonomy  $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ , generated by  $\bar{t}, \bar{\beta}$ .  $Z(\pi) = \langle t^2 \rangle \cong \mathbb{Z}$ . We have  $gp\langle \alpha, \beta \rangle = \pi_1(K)$  and so  $\pi \cong \pi_1(K) \times \langle t \rangle$ .  $\pi_1(K)$  is characteristic.
10.  $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \alpha \beta^{-1} \rangle$  with holonomy  $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ , generated by  $\bar{t}, \bar{\beta}$ .  $Z(\pi) = \langle t^2 \rangle \cong \mathbb{Z}$ . We have  $gp\langle \alpha, \beta \rangle = \pi_1(K)$  and  $\pi = \pi_1(K) \times \langle t \rangle$ .  $\pi_1(K)$  is characteristic.

We now state the result concerning the  $R_\infty$ -property of these groups.

**Theorem 2.** *Let  $\pi = \pi_1(M)$  where  $M$  is a compact flat 3-manifold. Then  $\pi$  has the  $R_\infty$ -property if  $\pi$  is isomorphic to one of the groups (3), (4), (5), (7), (8), (9) or (10). In the case when  $\pi$  is isomorphic to the groups (1), (2), or (6),  $M$  admits self-homeomorphisms with finite Reidemeister numbers.*

We merely outline the method of proof here, referring the reader to relevant papers for detailed proofs. In case (1), the manifold  $M$  is a torus and the assertion is well-known.

For Cases (4) and (5), we use the exact sequence (2). In these cases,  $Z = Z(\pi)$  is characteristic. Since  $\pi$  projects onto a two dimensional crystallographic group  $\Lambda = \pi/Z(\pi)$ , which is isomorphic to  $G_4$  and  $G_6$  respectively, the  $R_\infty$ -property of  $\pi$  follows from the  $R_\infty$ -property of  $\Lambda$  by [GW3].

For Case (7),  $\pi = \pi_1(K) \times \mathbb{Z}$  where  $K$  is the Klein bottle. This group is known to have the  $R_\infty$ -property (see [GW2, Theorem 2.4]). For Cases (9) and (10),  $\pi \cong \pi_1(K) \rtimes \mathbb{Z}$  where  $\pi_1(K)$  is characteristic. For any  $\varphi \in \text{Aut}(\pi)$ , the induced automorphism  $\bar{\varphi}$  is either  $id_{\mathbb{Z}}$  or  $-id_{\mathbb{Z}}$ . The former case yields  $R(\varphi) = \infty$ . In the later case, the set of twisted conjugacy classes of  $\varphi'$  injects into the set of twisted conjugacy classes of  $\varphi$ , i.e.,  $\mathcal{R}(\varphi') \hookrightarrow \mathcal{R}(\varphi)$ . Since  $\pi_1(K)$  has the  $R_\infty$ -property, it follows that  $R(\varphi) = \infty$ .

For Cases (2), (3), (8) we consider  $\bar{\pi} = \Phi$  and  $\Gamma = \mathbb{Z}^3$ . For Case (8),  $\Phi = \mathbb{Z}_2$  and for Case (3),  $\Phi = \mathbb{Z}_3$ . In each of these two cases, we use the exact sequence (1). One can find a representative  $g \in \pi$  of a suitable element of  $\Phi$  such that  $R(\iota_g \circ \varphi') = \infty$  (see [GWZ1, §3.3 and §4.3]). Since  $\Gamma$  is characteristic and since  $\Phi$  is finite, it follows that  $R(\varphi) = R(\iota_g \circ \varphi) = \infty$ . For Case (2), an explicit automorphism  $\varphi$  was constructed with  $R(\varphi) < \infty$  (see §7 of [GWZ1]).

For the Case (6), consider an automorphism  $\varphi$  with restriction  $\varphi'$  on  $\Gamma$  of type II and IV' from [GWZ1, Table 4.1]. One can write down all the other three automorphisms using  $\theta_1, \theta_2, \theta_3$  given few lines below the table. All such lifts have Reidemeister number finite for suitable values of  $\epsilon, r, s, b$ . Choosing  $\epsilon = 1, r = 1, s = a = 0$

in the notation of [GWZ1, Table 4.1, §4.2] we obtain that  $\phi' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$  so that

$R(\phi') = |\det(I - \phi')| = |\det(I - \theta_i \phi')| = R(\theta_i \phi') = 2$ . Now by the addition formula, namely, [GW1, Lemma 2.1], it follows that  $R(\varphi) < \infty$ . Moreover, this group in Case (6) is the classical Hantzsche-Wendt group which is known not to have the  $R_\infty$ -property (see [DDP]). In [GW5], all crystallographic groups of rank 2 are classified in terms of the  $R_\infty$ -property. Similarly, the result in [DP] includes the full classification of all crystallographic groups of rank 3 in terms of the  $R_\infty$ -property, which certainly include the ten 3-dimensional flat manifolds. So the result can be obtained from [DP] after identifying explicitly the flat 3-manifolds as described in [GWZ1] with the ones as described in [DP].

### 3. GEOMETRIES $S^2 \times \mathbb{R}$ AND Sol

**3.1.  $S^2 \times \mathbb{R}$ -geometry.** In this case let us first analyze which of the fundamental groups have the  $R_\infty$ -property and which of the spaces have the  $R_\infty$ -property. For the first question, the groups involved are:

- (a) the fundamental group of  $S^2 \times S^1$ , which is  $\mathbb{Z}$ , and does not have the  $R_\infty$ -property;
- (b) the fundamental group of  $\mathbb{R}P^2 \times S^1$ , which is  $\mathbb{Z}_2 \times \mathbb{Z}$ , and does not have the  $R_\infty$ -property;
- (c) the fundamental group of  $S^2 \tilde{\times} S^1$ , which is  $\mathbb{Z}$ , and does not have the  $R_\infty$ -property;
- (d) the fundamental group of  $\mathbb{R}P^3 \# \mathbb{R}P^3$ , which is  $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ , and it has the  $R_\infty$ -property.

Now we consider the question at the level of spaces. Certainly, since  $\pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3)$  has the  $R_\infty$ -property, the manifold  $\mathbb{R}P^3 \# \mathbb{R}P^3$  has the  $R_\infty$ -property.

For the other three manifolds,  $M = S^2 \times S^1$ ,  $\mathbb{R}P^2 \times S^1$ , and  $S^2 \tilde{\times} S^1$ , observe that in all these cases, they are total spaces of fibre bundles over  $S^1$ . In each case, it is easy to construct a fibre preserving map  $f : M \rightarrow M$  which induces the reflection map on the base space  $S^1$ . This implies that the induced map  $f_\#$  has finite Reidemeister number.

For more details and further results about the Nielsen theory of selfmaps on such manifolds see [GWZ2].

**3.2. Sol-geometry.** Let  $M$  be a Sol 3-manifold. Then  $M$  is one of the two types:

- (a) a mapping torus of a self-homeomorphism  $f : T \rightarrow T$  of the torus  $T = S^1 \times S^1$  which induces in  $\pi_1(T) = \mathbb{Z}^2$  an automorphism given by an Anosov matrix  $A \in GL(2, \mathbb{Z})$ ,
- (b) the union of two twisted  $I$ -bundles over the Klein bottle glued along their common boundaries, which are tori, via an Anosov diffeomorphism. Such a manifold is also known as a sapphire manifold.

**The case of a torus bundle.** The homeomorphism type of  $M$  is determined by the conjugacy class of  $A$  in  $GL(2, \mathbb{Z})$ . Thus  $M$  fibres over a circle with fibre  $T$  and we have an isomorphism  $G := \pi_1(M) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ . It turns out that the normal subgroup  $N := \mathbb{Z}^2 \subset G$  is characteristic (see [GW1, Lemma 2.1]). There are several cases to consider depending on the conjugacy class of  $A$ . Note that since  $A$  is Anosov, (i.e.,  $|Tr(A)| > 2$ ,  $\det A = \pm 1$ ), the eigenvalues of  $A$  are real and neither of them equals  $\pm 1$ . In particular  $A$  has infinite order.

Case (a): If  $\det A = -1$ , it was shown in [GW4] that any automorphism of  $G$  induces the identity map of the quotient  $G/N = \mathbb{Z}$ . Therefore  $G$  has the  $R_\infty$ -property in this case.

Case (b): We now assume that  $\det A = 1$ . Examples of  $A$  such that  $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$  does not have the  $R_\infty$ -property were given in [GW1, Example 4.3]. It was shown that when

$$(4) \quad A = \begin{pmatrix} k^2 + 1 & k \\ k & 1 \end{pmatrix},$$

$G$  does not have the  $R_\infty$ -property.

A necessary condition for an automorphism  $\phi : G \rightarrow G$  to have finite Reidemeister number is that the induced automorphism  $\bar{\phi}$  on  $G/N \cong \mathbb{Z}$  equals  $-id$ . Let  $S$  be the matrix of  $\phi|_N$ . Then  $\bar{\phi} = -id$  if and only if  $SA = A^{-1}S$ . Conversely, if  $S \in GL(2, \mathbb{Z})$  is such that  $SAS^{-1} = A^{-1}$ , then we obtain an automorphism  $\phi$  of  $G$  such that  $\phi|_N$  is given by  $S$  and  $\bar{\phi} = -id$ . *In particular, if  $A$  and  $A^{-1}$  are not conjugates in  $GL(2, \mathbb{Z})$ , then  $G$  has the  $R_\infty$ -property.*

From [GM1, Proposition 5,8, Theorem 5.9] we obtain that, if  $A, A^{-1} \in SL(2, \mathbb{Z})$  are conjugates in  $GL(2, \mathbb{Z})$ , then  $A$  must be conjugate to a matrix of the form  $A_0, B_0$ , or  $C_0$  where

$$(5) \quad A_0 = \begin{pmatrix} r & s \\ s & u \end{pmatrix}, B_0 = \begin{pmatrix} r & s \\ t & r \end{pmatrix}, \text{ and } C_0 = \begin{pmatrix} r & s \\ u-r & u \end{pmatrix}.$$

We remark that an  $A \in SL(2, \mathbb{Z})$  may be conjugate to more than one of the above three types.

**Type  $A_0$ :** *If  $A$  is conjugate to a matrix of the form  $A_0$  in Equation (5), then there is an automorphism of the group  $G = \pi_1(M)$  which has finite Reidemeister number.*

*Proof.* We may (and do) assume that  $A = A_0$ . Consider the automorphism of  $N = \mathbb{Z}^2$  given by the matrix  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Since  $JA_0J^{-1} = A_0^{-1}$ ,  $J$  extends to an automorphism  $\phi$  of the group  $G = N \rtimes_{A_0} \mathbb{Z}$ . Then  $\phi$  induces  $-id$  on the quotient  $G/N = \mathbb{Z}$ . The Reidemeister number  $R(\phi)$  can then be calculated using a certain *addition formula* (see [GW1, Lemma 2.1]). In our context, we obtain  $R(\phi) = R(J) + R(JA)$ . Since for any  $S \in GL(2, \mathbb{Z})$  we have  $R(S) = |\det(I - S)|$  when  $\det(S)$  is non-zero, we obtain that  $R(\phi) = |\det(I - J)| + |\det(I - A_0J)| = 4$ .  $\square$

It was shown in [GW4, Theorem 2.2] that the Nielsen number of any homeomorphism of  $M$  (where the gluing torus homeomorphism corresponds to  $A_0$ ) is equal to either 0 or 4 and that both possibilities do occur.

We shall now treat simultaneously the remaining cases when  $A$  is of type  $B_0$  or  $C_0$ . Recall that  $\det(A) = 1$ . We say that an element  $A_1 \in GL(2, \mathbb{Z})$  is a *primitive root* of  $A$  if  $\delta A = A_1^m$  with  $m \geq 1$  maximum and  $\delta \in \{1, -1\}$ .

**Types  $B_0$  and  $C_0$ :** *Let  $A$  be conjugate to an Anosov matrix of the form  $B_0$  or  $C_0$  in Equation (5). Then: the group  $G = \pi_1(M) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$  has the  $R_\infty$ -property if and only if any primitive root of  $A$  has determinant  $+1$ .*

*Proof.* We assume, as we may, that  $A = \begin{pmatrix} r & s \\ t & r \end{pmatrix}$  when it is of type  $B_0$  and  $A = \begin{pmatrix} r & s \\ u-r & r \end{pmatrix}$  when



it is of type  $C_0$ . Also, since the case when  $A$  is of type  $A_0$  had already been considered, we assume that  $t \neq s$ .

Let  $\phi \in \text{Aut}(G)$ . Let  $S \in GL(2, \mathbb{Z})$  be the matrix of the automorphism  $\phi|_N$ , where  $N = \mathbb{Z}^2$ . Recall that  $N$  is characteristic in  $G$ . If  $\bar{\phi}$  induces the identity on  $G/N \cong \mathbb{Z}$ , then  $R(\phi) = \infty$ . So assume that  $\bar{\phi} = -id$ . Then  $S$  satisfies the equation  $SAS^{-1} = A^{-1}$ .

Consider the group  $K(A) = \{X \in SL(2, \mathbb{Z}) \mid XAX^{-1} = \delta A, \delta \in \{1, -1\}\}$ . Note that the centralizer  $Z(A) \subset SL(2, \mathbb{Z})$  is subgroup of  $K(A)$  of index at most 2. Since  $A \in SL(2, \mathbb{Z})$  is Anosov, it follows that its centralizer  $Z(A) \subset GL(2, \mathbb{Z})$  is virtually infinite cyclic. In fact, the image of  $K(A)$  in  $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I_2, -I_2\}$  under the natural projection equals the centralizer  $Z(\bar{A})$  where  $\bar{A} := \{\pm A\} \in PSL(2, \mathbb{Z})$ . Using the length function associated to the free product  $PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ , we see that  $\bar{A}$  has a *unique* primitive root  $\bar{A}_1$  in  $PSL(2, \mathbb{Z})$ . That is, there exists a unique  $\bar{A}_1 \in PSL(2, \mathbb{Z})$  such that  $\bar{A} = \bar{A}_1^k$  with  $k \geq 1$  largest. Moreover  $Z(\bar{A})$  equals  $\langle \bar{A}_1 \rangle \cong \mathbb{Z}$ . Then  $A_1 \in K(A)$  is such that  $A_1^k = \delta_0 A$ ,  $\delta_0 \in \{1, -1\}$ . The only other element in  $SL(2, \mathbb{Z})$  to have this property in  $K(A)$  is  $-A_1$ . Further,  $K(A)$  equals  $gp\langle A_1, -I_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$ .

*"If" part:* Suppose that every primitive root  $A'$  of  $A$  in  $GL(2, \mathbb{Z})$  has determinant  $+1$ . Then  $A' = A_1$  or  $-A_1$ . We set  $S_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  when  $A$  is of type  $B_0$  and  $S_0 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$  when  $A$  is of type  $C_0$ . Note that  $S_0^2 = I_2$  and  $S_0 A S_0^{-1} = A^{-1}$  in each type. We shall show that the same holds for the primitive root  $A_1$ .

Suppose that  $YAY^{-1} = \delta A^\epsilon$ ,  $\delta, \epsilon \in \{1, -1\}$ . Then  $S_0 Y A Y^{-1} S_0^{-1} = \delta S_0 A^\epsilon S_0^{-1} = \delta A^{-\epsilon}$ . Hence the group  $N(A) := \{S \in GL(2, \mathbb{Z}) \mid SAS^{-1} = \delta A^\epsilon, \delta, \epsilon \in \{1, -1\}\}$  contains  $K(A)$  as an index-2 subgroup since  $A_1$  is a primitive root of  $A$  in  $GL(2, \mathbb{Z})$ . Moreover, since  $\pm A_1$  are the only primitive roots of  $A$  in  $GL(2, \mathbb{Z})$ , we have  $N(A) = K(A) \rtimes \langle S_0 \rangle$  and so, any element of  $N(A)$  can be expressed as  $\epsilon A_1^p S_0^q$ ,  $\epsilon \in \{1, -1\}$ ,  $p \in \mathbb{Z}$ ,  $q = 0, 1$ . Thus  $N(A)$  acts on  $K(A)$  via conjugation and also on  $Z(\bar{A})$ . In view of the uniqueness of the primitive root of  $\bar{A}$ , we see that,  $SA_1 S^{-1} = \delta_1 A_1^\epsilon$  for all  $S \in N(A)$ , where  $\delta_1, \epsilon \in \{1, -1\}$ . We have  $A^{-1} = S_0 A S_0^{-1} = S_0 A_1^k S_0^{-1} = \delta_1^k A^\epsilon$ . This implies that  $\epsilon = -1$  and  $\delta_1^k = 1$ . If  $k$  is odd, we have  $\delta_1 = 1$  and  $S_0 A_1 S_0^{-1} = A_1^{-1}$  and so  $A_1$  is of type  $B_0$  (resp.  $C_0$ ) depending on the matrix  $S_0$ . Since  $A_1^k = \delta_0 A$ , if  $k$  is odd, we replace  $A_1$  by  $-A_1$  so that  $(-A_1)^k = A$  resulting in  $\delta_0 = 1$ . Again  $-A$  is of type  $B_0$  (resp. type  $C_0$ ) and the same holds for  $A_1$  as well. It can be shown (by induction) that if  $X$  is of type  $B_0$  (resp.  $C_0$ ), the same is true of  $X^p$  for all non-zero integers  $p$ . Therefore if  $SAS^{-1} = A^{-1}$ , then  $S, AS \in N(A)$  are of the form  $\eta A_1^q S_0$  for some  $\eta \in \{1, -1\}$ ,  $q \in \mathbb{Z}$ .

To complete the proof that  $R(\phi) = \infty$ , we now apply the addition formula  $R(\phi) = R(S) + R(AS)$  where  $SAS^{-1} = A^{-1}$ . First consider the case when  $A$  is of type  $B_0$ . We have  $S = \eta A_1^p S_0$ ,  $\eta \in \{1, -1\}$  where  $S_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . If  $p = 0$ , then  $S = \eta S_0$  and  $I - \eta S_0$  is singular. So  $R(S) = \infty$ . Suppose that  $p \neq 0$ . Write  $A_1 = \begin{pmatrix} x & y \\ z & x \end{pmatrix} \in SL(2, \mathbb{Z})$ . Then  $S = \eta \begin{pmatrix} x & -y \\ z & -x \end{pmatrix}$ . So  $\det(I_2 - S) = 1 - (x^2 - yz) = 0$  since  $A_1 \in SL(2, \mathbb{Z})$ . Therefore  $R(\phi) = \infty$ . The same argument applies in the case of Type  $C_0$  to yield  $R(S) = \infty$  and so  $R(\phi) = \infty$ .

“Only if” part: Suppose that  $\delta A = X_0^m$  for some  $X_0 \in GL(2, \mathbb{Z})$ ,  $m \geq 1$ , with  $\det(X_0) = -1$ ,  $\delta \in \{1, -1\}$ . Then  $m$  is even; write  $m = 2n$ . Let  $X_1 = X_0^2$  so that  $X_1^n = \delta A$ . By what has been shown already,  $X_1$  is of type  $B_0$  or  $C_0$ . We claim that  $X_0$  is of the same type as  $A$ — $B_0$  or  $C_0$ . To see this, write  $X_0 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ . Then  $X_1^2 = \begin{pmatrix} p^2+qr & q(p+s) \\ r(p+s) & s^2+qr \end{pmatrix}$ . If  $A$  is of type  $B_0$ , then we must have  $p^2 = q^2$ . If  $p = -q$ , then  $X_1$ —and hence  $A$ —would be diagonal. So  $p = q$  and  $X_0$  is of type  $B_0$ . The proof that  $X_0$  is of type  $C_0$  when  $A$  is, is similar and omitted.

Consider the automorphism  $\phi$  of  $G$  whose restriction to  $N = \mathbb{Z}^2$  is given by  $S = S_0 X_0$ . Then  $SAS^{-1} = S_0 A S_0^{-1} = A^{-1}$ . So  $\bar{\phi} = -id$ . Using the addition formula we obtain that  $R(\phi) = R(S) + R(AS)$ . Proceeding as before, we see that  $R(S) = 2 = R(AS)$ .

In summary, we have shown that: *The group  $\pi_1(M) = G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$  has the  $R_\infty$ -property if (i)  $\det A = -1$ , (ii)  $A \in SL(2, \mathbb{Z})$  is not conjugate to  $A^{-1}$ , and, (iii)  $A \in SL(2, \mathbb{Z})$  is of the type  $B_0$  or  $C_0$ , and, neither  $A$  nor  $-A$  is in the cyclic group generated by an element of  $GL(2, \mathbb{Z})$  having determinant equal to  $-1$ . If  $A \in SL(2, \mathbb{Z})$  is of type  $A_0$ , then  $\pi_1(M)$  does not have the  $R_\infty$ -property.*

This completes the proof of part (d) of Main Theorem for torus bundles case.

**The case of a sapphire manifold.** Recall that a *sapphire* is a 3-manifold obtained from two orientable 3-manifolds which are twisted  $I$ -bundles over a Klein bottle glued along their boundary tori. A sapphire which is not a torus bundle over the circle admits Sol-geometry when the gluing map is an Anosov homeomorphism. If  $M$  is a sapphire which is not a torus bundle, it is double covered by a torus bundle  $\tilde{M}$  with fundamental group  $L := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$  where  $A$  is a hyperbolic matrix in  $SL(2, \mathbb{Z})$ . Moreover the index 2 subgroup  $L$  is characteristic in  $G := \pi_1(M)$  (see [GW4, Lemma 3.1]). So, any automorphism  $\phi$  of  $G$  restricts to an automorphism  $\phi'$  of  $L$ . It follows that  $R(\phi) = \infty$  if  $R(\phi') = \infty$ . If  $\phi'$  induces  $id$  on the quotient  $L/N \cong \mathbb{Z}$  where  $N \cong \mathbb{Z}^2$  is the characteristic subgroup of  $L$  corresponding to the fundamental group of the torus fibre in  $\tilde{M}$ , then  $R(\phi') = \infty$ . In case  $\phi'$  induces  $-id$ , choose an element  $\alpha \in G \setminus L$ . Denote by  $\iota_\alpha$  the inner conjugation by  $\alpha$ . Then the automorphism  $\iota_\alpha \circ \phi =: \psi$  has the same Reidemeister number as  $\phi$ . Moreover, using the fact that  $M$  does not admit an orientation reversing homeomorphism, it can be shown that, when  $\phi'$  induces  $-id$  on  $L/N$ , then  $\iota_\alpha \circ \phi'$  induces  $id$  on  $\phi$ . Again we are led to the conclusion that  $R(\phi) = \infty$ . See [GW4, Theorems 3.4 and 4.2] for a more geometric proof.

#### 4. GEOMETRIES $\mathbb{H}^2 \times \mathbb{R}$ AND $\widetilde{SL}(2, \mathbb{R})$

In this section, we focus on those geometric 3-manifolds that are finitely covered by 3-manifolds that are  $S^1$ -bundles over hyperbolic surfaces.

Let  $\Gamma = \pi_1(M)$  where  $M$  admits either a  $\mathbb{H}^2 \times \mathbb{R}$ -geometry or an  $\widetilde{SL}(2, \mathbb{R})$ -geometry. Then  $M$  admits a finite cover  $\hat{M} \rightarrow M$  such that  $\hat{M}$  fibres over an orientable finite volume hyperbolic surface  $\Sigma$  with fibre  $S^1$ . Thus  $\chi(\Sigma) < 0$  and  $\Sigma$  is compact if and only if  $M$  is.

In the case of  $\mathbb{H}^2 \times \mathbb{R}$ -geometry, the  $S^1$ -bundle may be assumed to be the product bundle  $\Sigma \times S^1$ .

Thus we have an exact sequence

$$1 \rightarrow Z \rightarrow \Lambda \xrightarrow{\eta} \pi_1(\Sigma) \rightarrow 1 \quad (2)$$

where  $Z = \pi_1(S^1) \cong \mathbb{Z}$ . In the case of  $\mathbb{H}^2 \times \mathbb{R}$ -geometry, since  $\hat{M} \cong \Sigma \times S^1$ ,  $Z$  equals the centre of  $\Lambda$ .

Suppose that  $M$  has the  $\widetilde{SL(2, \mathbb{R})}$ -geometry. Let  $\mathcal{Z}$  denote the centre of  $\Lambda$ . Since  $\pi_1(\Sigma)$  is a nonabelian free group or a higher genus surface group, its centre is trivial. It follows that  $\mathcal{Z} \subset Z$ . We claim that  $\mathcal{Z}$  is non-trivial and hence infinite cyclic. To get a contradiction, suppose that  $\mathcal{Z}$  is trivial. Then  $\Lambda$  maps isomorphically onto its image under the projection  $p : \widetilde{SL(2, \mathbb{R})} \rightarrow PSL(2, \mathbb{R})$  in view of the fact that  $\ker(p) \cong \mathbb{Z}$  is the centre of  $\widetilde{SL(2, \mathbb{R})}$ . So  $\hat{M} = \widetilde{SL(2, \mathbb{R})}/\Lambda \rightarrow PSL(2, \mathbb{R})/p(\Lambda)$  is an infinite covering projection. This contradicts the finiteness of the volume of  $\hat{M}$ . Hence our claim.

**Theorem 3.** *Let  $M$  be a 3-manifold which admits a  $\mathbb{H}^2 \times \mathbb{R}$ -geometry or  $\widetilde{SL(2, \mathbb{R})}$ -geometry. Then  $\pi_1(M)$  has the  $R_\infty$ -property.*

*Proof.* By the above discussion, the group  $\Gamma = \pi_1(M)$  has a finite index subgroup  $\Gamma_0$  whose centre  $\mathcal{Z}$  is an infinite cyclic group and  $\Gamma_0/\mathcal{Z}$  is either a nonabelian free group of finite rank or the fundamental group of a closed surface  $\Sigma_0$  of genus  $g \geq 2$ . Thus  $\Gamma_0/\mathcal{Z}$  is a non-elementary hyperbolic group and so has the  $R_\infty$ -property. Hence, it follows from [GW2, Lemma 1.1] that  $\Gamma_0$  has the  $R_\infty$ -property. The same argument shows that any finite index subgroup of  $\Gamma_0$  also has the  $R_\infty$ -property.

Since  $\Gamma$  is finitely generated and since  $\Gamma_0$  has finite index in  $\Gamma$ , it follows that there is a finite index subgroup  $K \subset \Gamma_0$  that is characteristic in  $\Gamma$ . (For example, we may take  $K$  to be the intersection of all subgroups of  $\Gamma$  having index equal to the index of  $\Gamma_0$  in  $\Gamma$ .) Then  $K$  has the  $R_\infty$ -property and so, by [GW2, Lemma 1.1],  $\Gamma$  also has the  $R_\infty$ -property.  $\square$

## 5. NIL-GEOMETRY

The closed 3-manifolds which admit Nil-geometry are listed as the infranilmanifolds  $M$  of dimension 3, following Dekimpe [De, Theorem 6.5.5, Chapter 6, p. 154]. The last column denotes the holonomy group  $F$ . The group  $F$  may be described as the  $\pi_1(M)/\Gamma$  where  $\Gamma$  is the unique maximal nilpotent normal subgroup of  $\pi_1(M)$ . The manifold  $M$  is then covered by the compact nilmanifold  $\tilde{M}$  with covering group  $F$ . Since  $M$  has Nil-geometry, it is understood that  $\Gamma$  is not abelian; equivalently  $\tilde{M}$  is not the torus.

Type	Set of Seifert Invariants	$F$
i	$M_1(k) = \{k, (o_1, 1); \}$	1
ii	$M_2(k) = \{k - 2, (o_1, 0); (2, 1), (2, 1), (2, 1), (2, 1)\}$	$\mathbb{Z}_2$
iii	$M_3(k) = \{k, (n_2, 2); \}$	$\mathbb{Z}_2$
iv	$M_4(k) = \{k - 1, (n_2, 1); (2, 1), (2, 1)\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
v	$M_5(k) = \{k - 2, (o_1, 0); (4, 3), (4, 3), (2, 1)\}$ $M_6(k) = \{k - 1, (o_1, 0); (4, 1), (4, 1), (2, 1)\}$ $M_7(k) = \{k - 2, (o_1, 0); (4, 3), (4, 1), (2, 1)\}$	$\mathbb{Z}_4$
vi	$M_8(k) = \{k - 2, (o_1, 0); (3, 2), (3, 2), (3, 2)\}$ $M_9(k) = \{k - 1, (o_1, 0); (3, 1), (3, 1), (3, 1)\}$ $M_{10}(k) = \{k - 2, (o_1, 0); (3, 2), (3, 1), (3, 1)\}$ $M_{11}(k) = \{k - 2, (o_1, 0); (3, 2), (3, 2), (3, 1)\}$	$\mathbb{Z}_3$
vii	$M_{12}(k) = \{k - 2, (o_1, 0); (6, 5), (3, 2), (2, 1)\}$ $M_{13}(k) = \{k - 1, (o_1, 0); (6, 1), (3, 1), (2, 1)\}$ $M_{14}(k) = \{k - 2, (o_1, 0); (6, 1), (3, 2), (2, 1)\}$ $M_{15}(k) = \{k - 2, (o_1, 0); (6, 5), (3, 1), (2, 1)\}$	$\mathbb{Z}_6$

In this table the integer  $k$  is assumed to be strictly bigger than 0. The case of manifolds with the same Seifert invariants but with  $k \leq 0$  are either flat manifolds or they represent manifolds which are already homeomorphic to one in the Table for  $k > 0$ . When the manifold is not flat, one passes from one family of invariants with  $k < 0$  to another with  $k > 0$  by changing the orientation (see [Se]).

The complete answer to the question about  $R_\infty$ -property of the manifolds above is given by [DP, Theorem 4.4] which in terms of the table above says: A closed infranil-manifold has the  $R_\infty$ -property, if and only if it does not belong to the first two lines. Furthermore, for the cases of the manifolds of types (i) and (ii), namely the ones which do not have the  $R_\infty$ -property, the Reidemeister spectrum is described in [Te, section 5].

**Remark 4.** *The fundamental groups of the Seifert manifolds have a presentation given by [Or, Chapter 5 section 5.3]. It follows from such presentation that we have a short exact sequence  $1 \rightarrow \langle h \rangle \rightarrow \pi_1(M) \rightarrow \pi_1(B) \rightarrow 1$ , where  $h$  is a generator which corresponds to the regular fibre. From [Or, Chapter 5, §5.3, Lemma 1], it follows that this short exact sequence is characteristic with respect to automorphisms of  $\pi_1(M)$ . The results obtained for the 3-manifolds may help to study the  $R_\infty$ -property for the groups  $\pi_1(B)$ . The groups  $\pi_1(B)$  are most often Fuchsian groups.*

## 6. APPENDIX

The purpose of the appendix is to show that the notions of a space  $X$  having the  $R_\infty$ -property and the group  $\pi_1(X)$  having the  $R_\infty$ -property are not equivalent.

Certainly if  $\pi_1(X)$  has the  $R_\infty$ -property, then the space  $X$  has the  $R_\infty$ -property. We now provide an example where  $X$  has the  $R_\infty$ -property but its fundamental group  $\pi_1(X)$  does not have.

Let  $Y = S^3 \times S^3$  and  $h : Y \rightarrow Y$  a homeomorphism which induces on the homology group  $H_3(S^3 \times S^3; \mathbb{Z})$  a homomorphism given by a matrix

$$(6) \quad A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

in  $GL(2, \mathbb{Z})$ .

Let  $X$  be the mapping torus of the homeomorphism  $h$  of  $S^3 \times S^3$ , so  $X$  fibres over the circle  $S^1$  with fibre  $S^3 \times S^3$ . Let  $f : X \rightarrow X$  be a homotopy equivalence. Then the map  $f$  can be deformed to a fibre-preserving map. We assume that  $f$  itself is fibre preserving. The induced map  $\bar{f}$  on the base  $S^1$  is either of degree 1 or  $-1$ . The induced isomorphism on  $H_3(X) \cong \mathbb{Z}^2$  is a matrix  $B$  such that: If the degree of  $\bar{f}$  is  $-1$  then we must have  $BAB^{-1} = A^{-1}$ . Let  $A$  be a matrix where such  $B$  does not exist (see §3.2 above). Then any homotopy equivalence will induce a map of degree 1 on  $S^1$  and hence identity on  $\pi_1(X)$ . So  $R(f) = \infty$ , but certainly  $\mathbb{Z} = \pi_1(X) = \pi_1(S^1)$  does not have the  $R_\infty$ -property. So it suffices to choose a matrix  $A$  where  $B$  does not exist and determine a homeomorphism  $h_A$ . We can take  $A$  any matrix such that  $\det A = -1$  or one of the matrices from [GW1, Example 4.4]. In details, for the former case let

$$(7) \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

so that

$$(8) \quad A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Define  $h_A(q_1, q_2) = (q_1q_2, q_1^2q_2)$  and  $h_{A^{-1}}(q_1, q_2) = (q_2q_1^{-1}, q_1q_2^{-1}q_1)$ . It is straightforward to verify that  $h_A \circ h_{A^{-1}} = h_{A^{-1}} \circ h_A = Id_{S^3 \times S^3}$ . So we can apply our construction for  $h_A$  and we obtain the result.

For a matrix in the latter case (as in [GW1, Example 4.4]), let

$$(9) \quad A = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$$

so that

$$(10) \quad A^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}.$$

Define  $h_A(q_1, q_2) = (q_1^4q_2, q_1^3q_2)$  and  $h_{A^{-1}}(q_1, q_2) = (q_1q_2^{-1}, q_2q_1^{-1}q_2q_1^{-1}q_2q_1^{-1}q_2)$ . It is straightforward to verify that  $h_A \circ h_{A^{-1}} = h_{A^{-1}} \circ h_A = Id_{S^3 \times S^3}$ . So we can apply our construction for  $h_A$  and we obtain the result.

**Acknowledgments:** The authors grateful to the referee for his/her many valuable comments and for drawing our attention to some inaccuracies in an earlier version of this paper.

## REFERENCES

- [AFW] M. Aschenbrenner, S. Friedl and H. Wilton: 3-Manifold Groups, *Arxiv:1205.0202v3*, 24 Apr 2013.
- [De] K. Dekimpe: *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics, **1639**, Springer-Verlag, New York, 1996.
- [DDP] K. Dekimpe, B. De Rock, and P. Penninckx: The  $R_\infty$ -property for infra-nilmanifolds, *Topol. Methods Nonlinear Anal.*, **34** no. 2, 353–373, 2009.
- [DP] K. Dekimpe and P. Penninckx: Erratum to: The finiteness of the Reidemeister number of morphisms between almost-crystallographic groups *J. Fixed Point Theory Appl.*, **12** no. 1–2, 261–288, 2012.
- [F] A. Fel’shtyn, New directions in Nielsen-Reidemeister theory *Topology Appl.* **157** no. 10–11, 1724–1735, 2010.
- [GM1] D. L. Gonçalves and S. T. Martins: The group Aut and Out of the fundamental group of a closed Sol 3-manifold, *arXiv:1909.05292v1*, Sept. 2019.
- [GSW] D. L. Gonçalves, P. Sankaran, and P. Wong: Twisted conjugacy in free products. *Comm. Algebra*, <https://doi.org/10.1080/00927872.2020.1751848>, published online: April 17, 2020.
- [GW1] D. L. Gonçalves and P. Wong: Twisted conjugacy classes in exponential growth groups, *Bull. London Math. Soc.*, **35** no. 2, 261–268, 2003.
- [GW2] D. Gonçalves and P. Wong: Twisted conjugacy classes in nilpotent groups, *J. Reine Angew. Math.* **633**, 11–27, 2009.
- [GW3] D. L. Gonçalves; P. Wong: Twisted conjugacy for virtually cyclic groups and crystallographic groups, *Combinatorial and geometric group theory*, 119–147, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.
- [GW4] D. L. Gonçalves; P. Wong: Nielsen numbers of selfmaps of Sol 3-manifolds, *Topology Appl.* **159** no. 18, 3729–3737, 2012.
- [GW5] D. L. Gonçalves and P. Wong: Automorphisms of the two dimensional crystallographic groups, *Comm. Algebra*, **42** no. 2, 909–931, 2014.
- [GWZ1] D. L. Gonçalves; P. Wong, and Xue Zhi Zhao: Nielsen numbers of selfmaps of flat 3-manifolds, *Bull. Belg. Math. Soc. Simon Stevin* **21** no. 2, 193–222, 2014.
- [GWZ2] D. L. Gonçalves, P. Wong, and Xue Zhi Zhao: Nielsen theory on 3-manifolds covered by  $S^2 \times R$ , *Acta Math. Sin. (Engl. Ser.)* **31** no. 4, 615–636, 2015.
- [GWZ3] D. L. Gonçalves, P. Wong, and Xue Zhi Zhao: Mapping degrees between spherical 3-manifolds, *Sbornik Math.*, **208** no. 10, 1449–1472, 2017.
- [LL] G. Levitt and M. Lustig: Most automorphisms of a hyperbolic group have very simple dynamics, *Ann. Sci. École Norm. Sup.*, **33** no. 4, 507–517, 2000.
- [L] R. Lyndon: *Groups and Geometry*. LMS Lecture Note Series 101, Cambridge University Press, 1985 (reprinted with corrections 1986).
- [MS1] T. Mubeena and P. Sankaran: Twisted conjugacy classes in lattices in semisimple Lie groups, *Transform. Groups* **19** no. 1, 159–169, 2014.
- [MS2] T. Mubeena and P. Sankaran: Twisted conjugacy and quasi-isometric rigidity of irreducible lattices in semisimple Lie groups. *Indian J. Pure Appl. Math.* **50** no. 2, 403–412, 2019.
- [Or] P. Orlik: *Seifert Manifolds*. Lecture Notes in Mathematics, **291**, Springer-Verlag, Berlin, 1972.
- [Se] H. Seifert: Topologie dreidimensionaler gefaserner Räume. *Acta Math.*, **60**(1), 147–238, 1933.
- [Te] S. Tertooy: The Reidemeister spectra of low dimensional almost-crystallographic groups, *Experimental Mathematics*, <https://doi.org/10.1080/10586458.2019.1636426>

- [Wi] Geometrization conjecture: <https://en.wikipedia.org/wiki/Geometrization-conjecture>  
[ZVC] H. Zieschang, E. Vogt, H. Coldewey: *Surfaces and Planar Discontinuous Groups*. Lecture Notes in Mathematics, **835**, Springer-Verlag, Berlin, 1980.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA DA UNIVERSIDADE DE SÃO PAULO, DEPARTAMENTO DE MATEMÁTICA, RUA DO MATÃO, 1010 CEP 05508-090, SÃO PAULO-SP, BRASIL

*E-mail address:* `dlgoncal@ime.usp.br`

CHENNAI MATHEMATICAL INSTITUTE , H1 SIPCOT IT PARK, SIRUSERI, KELAMBAKKAM, 603103, INDIA

*E-mail address:* `sankaran@cmi.ac.in`

DEPARTMENT OF MATHEMATICS, BATES COLLEGE, LEWISTON, MAINE, USA

*E-mail address:* `pwong@bates.edu`