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ON EXPONENT AND NILPOTENCY OF $[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{K}P^n)]$

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Abstract. We give estimations of the nilpotency class and the p-primary exponent of the total Cohen groups $[\Omega(\mathbb{S}^{r+1}),\Omega(X)]$ especially, when X is the projective space over $\mathbb{K}=\mathbb{R},\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the quaternionic skew $\mathbb{R}$-algebra.

Introduction

The so-called total Cohen group $[\Omega(\mathbb{S}^{r+1}),\Omega(X)]$ is of interest in classical homotopy theory so it is natural to consider the exponent of such a group. Indeed, a fundamental question is when this group is nilpotent as raised in [6]. In [10], the group $[\Omega(\mathbb{S}^2),\Omega(\mathbb{S}^2)]$ is shown to be Abelian. Subsequently, we studied the group structure of such groups in [11] and [12]. Furthermore, the exponent of $[\Omega(\mathbb{S}^{r+1}),\Omega(X)]$ was investigated in [13], in particular when $X=\mathbb{K}P^n$ where $\mathbb{K}=\mathbb{R},\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the quaternionic skew $\mathbb{R}$-algebra and $n\leq\infty$. The objective of this paper is to make use of explicit computation of the Whitehead products in the projective spaces $\mathbb{K}P^n$ to give information about the homotopy nilpotency of $\Omega(\mathbb{K}P^n)$ (Theorem 1.23, Theorem 1.25) and the exponent of $[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{H}P^n)]$ (Theorem 3.1, Theorem 3.4).

This paper is organized as follows. Section 1 examines known results on the homotopy nilpotency of projective spaces and of spheres. Explicit computations of the Whitehead products in projective spaces $\mathbb{K}P^n$ and of the Whitehead length $\text{W-long}\,(\mathbb{K}P^n)$ are given. Sections 2 and 3 make use of the results from Section 1 to

(i) obtain and improve previous estimates of the homotopy nilpotency $\text{nil}\,\Omega(\mathbb{K}P^n)$ of the loop space of projective spaces $\mathbb{K}P^n$ obtained by Ganea [8] and by Snaith [23];

(ii) estimate $\exp_p[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{H}P^n)]$ for the p-primary exponent and correct the estimation for the p-primary exponent of $\exp_p[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{H}P^\infty)]$ presented in [13].

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In Section 4, we improve a result of Kachi [19] about the Samelson product in the symplectic group $\mathrm{Sp}(m)$ and generalize a result of Arkowitz and Curjel [1].

1. Homotopy nilpotency and Whitehead products

Throughout this paper, all spaces are assumed to be connected, based and of the homotopy type of CW-complexes and all maps are based maps unless stated otherwise. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the (based) loop (resp. suspension) space of a space X and $[Y, X]$ for the set of (based) homotopy classes of maps $Y \to X$. Given a nilpotent space X and a prime $p \geq 2$, we write $X_{(p)}$ for its p-localization and $X_{(0)}$ for its rationalization. We do not distinguish notationally between a continuous map and its homotopy class and we use notations from the book by Toda [25]. An H-space X is called a *group-like space* if X satisfies all the axioms of groups up to homotopy (see e.g., [26]). Recall that a homotopy associative X-space always has a homotopy inverse.

Given a group-like space X, we write $c_{1,X} = \mathrm{id}_X : X \to X$ and $c_{2,X} : X \times X \to X$ for the commutator map of X. For an integer $n \geq 1$, let X^n be the n-th Cartesian power of X. Define inductively the $(n+1)$-fold commutator map $c_{n+1,X} : X^{n+1} \to X$ as the composition $c_{n+1,X} = c_{2,X} \circ (c_{1,X} \times c_{n,X})$ for $n \geq 2$. Write $T(X, \ldots, X)$ and $X^{\wedge n} = X^n / T(X, \ldots, X)$ for the n-th fold fat wedge and smash products, respectively. Since the inclusion map $T(X, \ldots, X) \hookrightarrow X^n$ is a cofibration, the map $c_{n,X}$ gives rise to the iterated Samelson product $\bar{c}_{n,X} : X^{\wedge n} \to X$ with $q_n \circ \bar{c}_{n,X} = c_{n,X}$ for the quotient map $q_n : X^n \to X^{\wedge n}$.

The *homotopy nilpotency class* of a group-like space X is the least n such that $c_{n+1,X} \simeq *$ and $c_{n,X} \not\simeq *$ or equivalently, $\bar{c}_{n+1,X} \simeq *$ and $\bar{c}_{n,X} \not\simeq *$. In this case, we write $\mathrm{nil}\, X = n$ and call the homotopy associative H-space X *homotopy nilpotent*. By [4, Theorem 2.7] one has $\mathrm{nil}\, X = \sup_{m \geq 1} \mathrm{nil}\, [X^m, X] = \sup_Y \mathrm{nil}\, [Y, X]$ for any group-like space X, where Y ranges over all topological spaces. Then, it is obvious that for group-like spaces $X_1, \ldots, X_m$ for $m \geq 1$, we have

$$\mathrm{nil}\, X_1 \times \cdots \times X_m = \max\{\mathrm{nil}\, X_1, \ldots, \mathrm{nil}\, X_m\}.$$

Note that for a group-like space X, $\mathrm{nil}\, X = 1$ if and only if X is homotopy commutative. Certainly, $\mathrm{nil}\, K(\pi, 1) = 1$ provided the abstract group π is Abelian and $\mathrm{nil}\, K(\pi, n) = 1$ for the Eilenberg-MacLane space $K(\pi, n)$ with $n \geq 2$. Since $\Omega K(\pi, 1) \simeq \pi$ (as discrete spaces), we see that $\Omega K(\pi, 1)$ is homotopy nilpotent if and only if the group π is nilpotent.

In particular, $\mathrm{nil}\,\mathbb{R}P^\infty = \mathrm{nil}\,\Omega(\mathbb{R}P^\infty) = 1 = \mathrm{nil}\,\Omega(\mathbb{C}P^\infty) = \mathrm{nil}\,\mathbb{C}P^\infty$ and by [21], we have $\mathrm{nil}\,\Omega(\mathbb{H}P^\infty) = \mathrm{nil}\,\mathbb{S}^3 = \mathrm{nil}\,SU(2) = 3$ for the special unitary group $SU(2)$.

For the wedge $\mathbb{S}^m \vee \mathbb{S}^n$ of two spheres, by Hilton [15], there is a non-trivial iterated Whitehead product of any weight. Therefore, $\mathrm{nil}\,\Omega(\mathbb{S}^m \vee \mathbb{S}^n) = \infty$. According to [8, Proposition 1.2], $\mathrm{nil}\ \Omega(\mathbb{S}^n) \leq 2$ for any odd n; $\mathrm{nil}\ \Omega(\mathbb{S}^n) = 1$ if and only if $n = 1, 3, 7$; $\mathrm{nil}\ \Omega(\mathbb{S}^2) = 2$.

For Whitehead products in spheres, we state the following facts from [13, Propositions 1.4, 1.7, 1.8] which will be useful in later sections.

Proposition 1.1. *Let ι_n be the identity map of the n-sphere $\mathbb{S}^n$, $\beta_i \in \pi_{m_i}(\mathbb{S}^n)$ with $i = 1, 2, 3$, and X be a space. Then*

(1) *$[\iota_n, \iota_n]$ has infinite order if n is even, is trivial if $n = 1, 3, 7$, and has order 2 otherwise;*
(2) *$[[\iota_n, \iota_n], \iota_n]$ is trivial if $n = 2$ or n is odd, otherwise it has order 3;*
(3) *if $n \geq 2$, then $3[[\beta_1, \beta_2], \beta_3] = 0$ and all iterated Whitehead products of weight ≥ 4 vanish;*
(4) *if n is odd, then $2[\beta_1, \beta_2] = 0$ and $[\beta_1, \beta_2, \beta_3] = 0$;*
(5) *if $\alpha \in \pi_k(X)$, $\beta \in \pi_l(X)$ and $[\alpha, \beta] = 0$ then $[\alpha \circ \delta, \beta \circ \delta'] = 0$ for $\delta \in \pi_s(\mathbb{S}^k)$ and $\delta' \in \pi_t(\mathbb{S}^l)$;*
(6) *(Jacobi identity) If $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $\gamma \in \pi_{r+1}(X)$, and $p, q, r > 0$, then*

$$(-1)^{(p+1)(r+1)}[[\alpha, \beta], \gamma] + (-1)^{(q+1)(p+1)}[[\beta, \gamma], \alpha] + (-1)^{(q+1)(r+1)}[[\gamma, \alpha], \beta] = 0;$$

(7) *if $\alpha \in \pi_k(X)$, $\beta \in \pi_l(X)$ and $\delta \in \pi_s(\mathbb{S}^{k-1})$, $\delta' \in \pi_t(\mathbb{S}^{l-1})$ then $[\alpha \circ \Sigma\delta, \beta \circ \Sigma\delta'] = [\alpha, \beta] \circ \Sigma(\delta \wedge \delta')$.*

Given a space X and $\beta_k \in \pi_{m_k}(X)$ with $m_k \geq 1$ for $k = 1, \ldots, n$, denote by $[\beta_1, \ldots, \beta_n]$ the iterated n-fold Whitehead product $[[\ldots[\beta_1, \beta_2], \ldots], \beta_n]$. The *Whitehead length* $\text{W-long}\,(X)$ [4, Definition 4.5] is the least integer $n \geq 0$ such that the iterated n-fold Whitehead product $[\beta_1, \ldots, \beta_n] = 0$ for all $\beta_k \in \pi_{m_k}(X)$ with $m_k \geq 1$ for $k = 1, \ldots, n$; if no such integer exists, $\text{W-long}\,(X) = \infty$. Then, by [4, Theorem 4.6], we have

$$\text{W-long}\,(X) \leq \mathrm{nil}\,\Omega(X). \tag{1.2}$$

Hence, (1.2) and Proposition 1.1 imply that $\mathrm{nil}\,\Omega(\mathbb{S}^n) = 1$ if and only if $n = 1, 3, 7$ and that

$$\mathrm{nil}\,\Omega(\mathbb{S}^n) \geq \begin{cases} 2 & \text{for odd } n \text{ and } n \neq 1, 3, 7 \text{ or } n = 2; \\ 3 & \text{for even } n \geq 4. \end{cases}$$

It is well known that the homotopy groups $\{\pi_n(X)\}_{n>1}$ of a space X can be turned into a graded quasi-Lie algebra with respect to the Whitehead product in X. At the end of this section, we plan to use the following result which, in the context of quasi-Lie algebras, is probably well known to experts in the Lie algebra theory.

Lemma 1.3. *Let X be a space. For any positive integer $m \geq 2$, any iterated Whitehead product of weight m can be written as a sum of the iterated m-fold Whitehead products of the form $[- \ , \ldots, \ -]$.*

Proof. The result is true for iterated Whitehead product of weight $m = 2$. Suppose that the result holds for iterated Whitehead products of weight $\leq N$.

Given an iterated Whitehead product of weight $N+1$, it is of the form $\beta = [\delta, \delta']$, where δ and δ' are iterated Whitehead products of weights k and l, respectively where $k+l = N+1$. By the inductive hypothesis and by the linearity of the Whitehead product, β is a linear combination of Whitehead products of the form $[-,-]$, where the entries are k-fold and l-fold iterated Whitehead products. Thus, without loss of generality, we may assume that $\delta = [[\ldots[\delta_1, \delta_2]\ldots], \delta_k]$, $\delta' = [[\ldots[\delta'_1, \delta'_2]\ldots], \delta'_l]$, and $k+l = N+1$. If $l = 1$, the result follows. Next, we induct on l. Suppose that the result holds for all values of $1 \leq l < K$ and let us show that the result holds for $l = K$.

Now, we write $\delta' = [\delta'', \delta'_K]$ where δ'' is an iterated $(K-1)$-fold Whitehead product. Then, $\beta = [\delta, \delta'] = [\delta, [\delta'', \delta'_K]]$. By Jacobi identity $\beta = \pm[\delta'', [\delta'_K, \delta]] \pm [\delta'_K, [\delta, \delta'']]$ and then by the anti-commutativity and the linearity, we have $\beta = \pm[[\delta'_K, \delta], \delta''] \pm [[\delta, \delta''], \delta'_K]$. Since δ'' has weight $K-1 < K$, by inductive hypothesis, each of the two summands of β is $(N+1)$-fold. Thus, β is $(N+1)$-fold and this completes the induction. □

[8, Remark 3.1] says "A different method may be used to prove that $\operatorname{nil}\Omega(\mathbb{S}^n) \leq 3$ for any even $n \geq 2$; this result is also known to M.G. Barratt and I. Berstein." For completeness, we present a proof of the following result in which the case when n is even is due to Stephen Theriault [24]. We should point out that the proof in the case when n is odd is similar to that in the case when n is even.

Proposition 1.4. $\operatorname{nil}\Omega(\mathbb{S}^n) \leq \begin{cases} 3 \text{ for } n \text{ even;} \\ 2 \text{ for } n \text{ odd.} \end{cases}$

Proof. (1): $n = 2m$. Notice that in view of [8, Proposition 1.2], we may assume that $m \geq 2$. Consider $\bar{c}_{2,\Omega\mathbb{S}^{2m}} : \Omega\mathbb{S}^{2m} \wedge \Omega\mathbb{S}^{2m} \to \Omega\mathbb{S}^{2m}$ and take its adjoint $\Sigma(\Omega\mathbb{S}^{2m} \wedge \Omega\mathbb{S}^{2m}) \to$

$\mathbb{S}^{2m}$ which is the Whitehead product $[ev_{2m}, ev_{2m}]$ of the evaluation map $ev_{2m} : \Sigma\Omega\mathbb{S}^{2m} \to \mathbb{S}^{2m}$ with itself. Now, since $\Sigma\Omega\mathbb{S}^{2m}$ is homotopy equivalent to $\Sigma(\bigvee_{k=1}^{\infty}(\mathbb{S}^{(2m-1)k}))$, we easily deduce that $\Sigma(\Omega\mathbb{S}^{2m} \wedge \Omega\mathbb{S}^{2m})$ is homotopy equivalent to the wedge $W = \Sigma(\bigvee_{k=1}^{\infty}(\mathbb{S}^{(2m-1)k})) \wedge (\bigvee_{k=1}^{\infty}(\mathbb{S}^{(2m-1)k}))$ of spheres, and up to a self-equivalence of W the map $[ev_{2m}, ev_{2m}]$ can be rewritten as $\theta_{2,\Omega(\mathbb{S}^{2m})} : W \to \mathbb{S}^{2m}$ where $\theta_{\Omega(\mathbb{S}^{2m}),2}$ is a wedge sum of iterated Whitehead products $\mathbb{S}^{(2m-1)k+1} \to \mathbb{S}^{2m}$ of the form $[\iota_{2m}, [\iota_{2m}, \ldots [\iota_{2m}, \iota_{2m}] \ldots]$, where k copies of ι_{2m} appear in the bracket.

Since the Whitehead product $[\iota_{2m}, [\iota_{2m}, [\iota_{2m}, \iota_{2m}]]] = 0$ but $[\iota_{2m}, [\iota_{2m}, \iota_{2m}]] \neq 0$, the map $\theta_{2,\Omega(\mathbb{S}^{2m})}$ above vanishes on all iterated brackets with four or more ι_{2m} provided $m \geq 2$. More generally, the adjoint of $\bar{c}_{k,\Omega\mathbb{S}^{2m}} : (\Omega\mathbb{S}^{2m})^{\wedge k} \to \Omega\mathbb{S}^{2m}$ for $k \geq 4$ can be rewritten in the same way and now each of the brackets appearing in $\theta_{k,\Omega\mathbb{S}^{2m}} : \Sigma((\Omega\mathbb{S}^{2m})^{\wedge k}) \to \mathbb{S}^{2m}$ involves at least four copies of ι_{2m}, so $\theta_{k,\Omega\mathbb{S}^{2m}} = 0$ for $m \geq 2$ and $k \geq 4$. Finally, we deduce $\operatorname{nil}\Omega(\mathbb{S}^{2m}) \leq 3$.

(2): If $n = 2m+1$ then $[\iota_{2m+1}, \iota_{2m+1}] = 0$ for $m = 0, 1, 3$ and $[\iota_{2m+1}, [\iota_{2m+1}, \iota_{2m+1}]] = 0$ but $[\iota_{2m+1}, \iota_{2m+1}] \neq 0$ provided $m \neq 0, 1, 3$. Then, the methods *mutatis mutandis* above yield $\operatorname{nil}\Omega(\mathbb{S}^{2m+1}) \leq 2$ and the proof is complete. $\square$

Notice that Proposition 1.4 yields

Corollary 1.5. *If* $n \geq 1$ *then* $\text{W-long}\,(\mathbb{S}^n) = \operatorname{nil}\Omega(\mathbb{S}^n)$.

Now let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the quaternionic skew $\mathbb{R}$-algebra and $\dim_{\mathbb{R}} \mathbb{K} = d$. Write $\mathbb{K}P^n$ for the nth projective space over $\mathbb{K}$, $i_{n,\mathbb{K}} : \mathbb{S}^d \hookrightarrow \mathbb{K}P^n$ for the inclusion of the bottom cell in $\mathbb{K}P^n$ and $\gamma_{n,\mathbb{K}} : \mathbb{S}^{d(n+1)-1} \to \mathbb{K}P^n$ for the quotient map. Recall that there is a fibration $\mathbb{S}^{d-1} \hookrightarrow \mathbb{S}^{d(n+1)-1} \longrightarrow \mathbb{K}P^n$. Since $d-1 < d(n+1)-1$ the inclusion $\mathbb{S}^{d-1} \hookrightarrow \mathbb{S}^{d(n+1)-1}$ is nullhomotopic. A result of Eckmann and Hilton [7] implies that there is a homotopy equivalence $\Omega(\mathbb{K}P^n) \simeq \mathbb{S}^{d-1} \times \Omega(\mathbb{S}^{d(n+1)-1})$. It follows that

$$\pi_k(\mathbb{K}P^n) = \gamma_{n,\mathbb{K}*}\pi_k(\mathbb{S}^{d(n+1)-1}) \oplus i_{n,\mathbb{K}*}\Sigma\pi_{k-1}(\mathbb{S}^{d-1}) \tag{1.6}$$

where, by abuse of notation, $\Sigma\pi_{k-1}(\mathbb{S}^{d-1})$ is the image of $\pi_{k-1}(\mathbb{S}^{d-1})$ under the suspension homomorphism $\Sigma : \pi_{k-1}(\mathbb{S}^{d-1}) \to \pi_k(\mathbb{S}^d)$. By means of [8, Theorem 1.1], we have

Theorem 1.7. *If* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *then* $\Omega(\mathbb{K}P^n)$ *has an* H*-homotopy type of* $\mathbb{S}^{d-1} \times \Omega(\mathbb{S}^{d(n+1)-1})$ *if and only if* $n \geq 3$ *is odd. The space* $\Omega(\mathbb{H}P^n)$ *has an* H*-homotopy type of* $\mathbb{S}^3 \times \Omega(\mathbb{S}^{4n+3})$ *if* $n \equiv -1 \pmod{24}$.

The well-known homotopy equivalence $\Sigma\Omega(\mathbb{S}^{d(n+1)-1}) \simeq \bigvee_{k=1}^{\infty} \Sigma(\mathbb{S}^{d(n+1)-2})^{\wedge k}$ leads to the second *Hopf-James* invariant $H_2 : \Omega(\mathbb{S}^{d(n+1)-1}) \to \Omega(\mathbb{S}^{2d(n+1)-3})$. We write

$$h_2 : \pi_m(\mathbb{S}^{d(n+1)-1}) \to \pi_m(\mathbb{S}^{2d(n+1)-3})$$

for the induced homomorphism of homotopy groups with $m \geq 1$.

To facilitate the computations involving Whitehead products in this section, we recall that one of the *exterior cup products* from [3, Chapter II] is the pairing

$$\underline{\sharp}\colon [\Sigma(X), \Sigma A] \times [\Sigma(Y), \Sigma B] \longrightarrow [\Sigma(X) \wedge Y, \Sigma(A) \wedge B]$$

defined by the composition

$$\beta_1 \underline{\sharp} \beta_2 \colon \Sigma(X) \wedge Y = X \wedge \Sigma(Y) \xrightarrow{X \wedge \beta_2} X \wedge \Sigma(B) = \Sigma(X) \wedge B \xrightarrow{\beta_1 \wedge B} \Sigma(A) \wedge B \tag{1.8}$$

for $(\beta_1, \beta_2) \in [\Sigma(X), \Sigma(A)] \times [\Sigma(Y), \Sigma(B)]$, where $\beta_1 \wedge B$ is the map $\beta_1 \wedge \mathrm{id}_B$ and $X \wedge \beta_2$ is the map $\mathrm{id}_X \wedge \beta_2$, up to the shuffle of the suspension coordinates.

Lemma 1.9. *If* $\beta \in \pi_{m-1}(\mathbb{S}^{d(n+1)-2})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{d(n+1)-1})$ *for* $k = 1, 2$, $d = 1$ *with* n *odd and* $d = 2, 4$ *with any* n *then*

(1) $h_2(\Sigma(\beta)) = 0$*;*

(2) $h_2([\beta_1, \beta_2]) = 0$.

Proof. (1): First, it is easily seen that the composite map

$$\mathbb{S}^{d(n+1)-2} \xrightarrow{\eta_{\mathbb{S}^{d(n+1)-2}}} \Omega(\mathbb{S}^{d(n+1)-1}) \xrightarrow{H_2} \Omega(\mathbb{S}^{2d(n+1)-3})$$

is null homotopic for the unit map $\eta_{\mathbb{S}^{d(n+1)-2}} : \mathbb{S}^{d(n+1)-2} \to \Omega(\mathbb{S}^{d(n+1)-1})$.

Together with the following commutative diagram

$$\begin{array}{ccc} \mathbb{S}^{m-1} & \xrightarrow{\eta_{\mathbb{S}^{m-1}}} & \Omega(\mathbb{S}^{m}) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \Omega\Sigma(\beta)} \\ \mathbb{S}^{d(n+1)-2} & \xrightarrow[\eta_{\mathbb{S}^{d(n+1)-2}}]{} & \Omega(\mathbb{S}^{d(n+1)-1}) \end{array}$$

associated with a given map $\Sigma(\beta) : \mathbb{S}^m \to \mathbb{S}^{d(n+1)-1}$, we can deduce that $h_2(\Sigma(\beta)) = 0$.

(2): Given $\beta_k \in \pi_{m_k}(\mathbb{S}^{d(n+1)-1})$ for $k = 1, 2$, $d = 1$ with n odd and $d = 2, 4$ with any n, by [3, (1.4) Proposition, Chapter IV], we get $[\beta_1, \beta_2] = [\iota_{d(n+1)-1}, \iota_{d(n+1)-1}](\beta_1 \underline{\sharp} \beta_2)$. But, in view of (1.8), we have $\beta_1 \underline{\sharp} \beta_2 = \Sigma^{d(n+1)-1}(\beta_2)\Sigma^{m_2-1}(\beta_1)$. Consequently,

$$[\beta_1, \beta_2] = [\iota_{d(n+1)-1}, \iota_{d(n+1)-1}]\Sigma^{d(n+1)-2}(\beta_2)\Sigma^{m_2-1}(\beta_1).$$

By [14, (1.90)], we have $[\iota_{d(n+1)-1}), \iota_{d(n+1)-1})] = \Sigma(\tau_{d(n-1)-2})$ for $d = 2, 4$ with any n and $[\iota_n, \iota_n] = \Sigma(\tau_{n-1})$ for $d = 1$ and n odd. Therefore, we obtain

$$[\beta_1, \beta_2] = \begin{cases} \Sigma(\tau_{n-1}\Sigma^{d(n+1)-3}(\beta_2)\Sigma^{m_2-2}(\beta_1)) \text{ for } d = 1 \text{ with } n \text{ odd,} \\ \Sigma(\tau_{d(n-1)-2}\Sigma^{d(n+1)-3}(\beta_2)\Sigma^{m_2-2}(\beta_1)) \text{ for } d = 2, 4 \text{ with any } n. \end{cases}$$

Then, (1) implies $h_2([\beta_1, \beta_2]) = 0$ and the proof is complete. □

Remark 1.10. Note that if $\beta_k \in \pi_{m_k}(\mathbb{S}^n)$ for $k = 1, 2$ with n odd then the proof of Lemma 1.9 yields $[\beta_1, \beta_2] = \Sigma(\beta)$ for some $\beta \in \pi_{m_1+m_2-2}(\mathbb{S}^{n-1})$.

Setting $\nu_n^+ = \nu_n + \alpha_1(n)$ for $n \geq 4$, in view of [2, (4.1-3)] and [14, Lemma 2.4], we have

Proposition 1.11. *If* $\beta \in \pi_m(\mathbb{S}^{d(n+1)-1})$ *then*

(1) $[\gamma_{n,\mathbb{R}}\beta, i_{n,\mathbb{R}}] = \begin{cases} 0 \textit{ for odd } n, \\ (-1)^m \gamma_{n,\mathbb{R}}(-2\beta + [\iota_n, \iota_n] \circ h_2(\beta)) \textit{ for even } n; \end{cases}$

(2) $[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}] = \begin{cases} 0 \textit{ for odd } n, \\ \gamma_{n,\mathbb{C}}(\eta_{2n+1} \circ \Sigma(\beta) + [\iota_{2n+1}, \eta_{2n+1}] \circ \Sigma h_2(\beta)) \textit{ for even } n; \end{cases}$

(3) $[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}] = \pm(n+1)\gamma_{n,\mathbb{H}}(\nu^+_{4n+3} \circ \Sigma^3(\beta) + [\iota_{4n+3}, \nu_{4n+3}] \circ \Sigma^3(h_2(\beta)))$.

Proposition 1.11(1) leads to

Corollary 1.12. (1) *If* $n \geq 2$ *is even then* W-long$(\mathbb{R}P^n) = \infty$ *for any even* n*;*

(2) W-long$(\mathbb{R}P^n) = 2$ *for any odd* n *unless* $n = 1, 3, 7$*;*

(3) W-long$(\mathbb{R}P^n) = 1$ *if and only if* $n = 1, 3, 7$.

Furthermore, if p *is a prime then*

(4) W-long$(\mathbb{R}P^n_{(p)}) = 1$ *for any odd* n *and* $p \geq 3$*;*

(5) W-long$(\mathbb{R}P^n_{(p)}) = 1$ *for* $n = 1, 3, 7$*;*

(6) W-long$(\mathbb{R}P^n_{(2)}) = 2$ *for any odd* n *unless* $n = 1, 3, 7$.

Proof. (1): First, notice that Proposition 1.11(1) implies that for the iterated Whitehead product of weight $(k+1)$ we have $[\ldots[\gamma_{n,\mathbb{R}}, i_{n,\mathbb{R}}], i_{n,\mathbb{R}}], \ldots], i_{n,\mathbb{R}}] = (-1)^{n+k}2^k\gamma_{n,\mathbb{R}} \neq 0$ for $k \geq 1$ provided n is even.

(2): If n is odd with $n \neq 1,3,7$ then $[\gamma_{n,\mathbb{R}}, \gamma_{n,\mathbb{R}}] \neq 0$ and, by Proposition 1.11(1), we have $[\gamma_{n,\mathbb{R}}\beta, i_{n\mathbb{R}}] = 0$ for any $\beta \in \pi_m(\mathbb{S}^n)$. Consequently, all triple Whitehead products in $\mathbb{R}P^n$ vanish.

(3): If $n = 1,3,7$ then $\mathbb{R}P^n$ is an H-space and so W-long$(\mathbb{R}P^n) = 1$. If $n \neq 1,3,7$ then $[\gamma_{n,\mathbb{R}}, \gamma_{n,\mathbb{R}}] = \gamma_{n,\mathbb{R}}[\iota_n, \iota_n] \neq 0$ and so W-long$(\mathbb{R}P^n) > 1$.

(4)-(6): Since the space $\mathbb{R}P^n$ is simple for any odd n, its p-localization $\mathbb{R}P^n_{(p)}$ exists. Then, the required formulae are direct consequences of (2)-(3) and the proof is complete. □

Now, we make use of Propositions 1.1 and 1.11 to examine the iterated Whitehead products in $\mathbb{K}P^n$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$. Due to the fact that any iterated Whitehead product of weight m can be expressed as a sum of iterated m-fold Whitehead products (Lemma 1.3), in what follows, we focus on iterated m-fold Whitehead products.

From equation (1.6) we have $\pi_m(\mathbb{C}P^n) = \gamma_{n,\mathbb{C}*}\pi_m(\mathbb{S}^{2n+1}) \oplus i_{n,\mathbb{C}*}\Sigma\pi_{m-1}(\mathbb{S}^1)$. Since, $\pi_{m'-1}(\mathbb{S}^1) = 0$ for $m' = 1$ or $m' \geq 3$, we only have to consider the suspension isomorphism $\Sigma : \pi_1(\mathbb{S}^1) \xrightarrow{\approx} \pi_2(\mathbb{S}^2)$.

Remark 1.13. In order to compute the double Whitehead product of two elements in $\mathbb{C}P^n$ it suffices to compute

(1) $[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2]$ for $\beta_k \in \pi_m(\mathbb{S}^{2n+1})$ with $k = 1,2$;

(2) $[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta]$ for $\beta \in \pi_m(\mathbb{S}^{2n+1})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$;

(3) $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2]$ for $\theta_l = \Sigma(\theta'_l) \in \pi_1(\mathbb{S}^1)$ with $l = 1,2$.

Furthermore,

(4) $[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta] = [\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}]\Sigma^m(\theta')$ for $\beta \in \pi_m(\mathbb{S}^{2n+1})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$;

(5) $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2] = [i_{n,\mathbb{C}}, i_{n,\mathbb{C}}]\Sigma(\theta'_1 \wedge \theta'_2)$ for $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1,2$.

Lemma 1.14. (1) *If $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ for $k = 1,2,3$ then*

(i) $2[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2] = 2\gamma_{n,\mathbb{C}}[\beta_1, \beta_2] = 0$*;*

(ii) $[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3] = \gamma_{n,\mathbb{C}}[[\beta_1, \beta_2], \beta_3] = 0$.

(2) *If* $n=1$ *then* $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2]\neq 0$ *and* $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2]=0$ *for* $n\geq 2$ *and* $\theta_l=\Sigma(\theta'_l)\in \Sigma\pi_1(\mathbb{S}^1)$ *with* $l=1,2$.

(3) $[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta]=0$ *for* $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2$ *and* $\theta=\Sigma(\theta')\in\Sigma\pi_1(\mathbb{S}^1)$.

(4) *If* $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *for* $k=1,2$ *and* $\theta=\Sigma(\theta')\in\Sigma\pi_1(\mathbb{S}^1)$ *then* $2[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2]=0$. *If* n *is odd then* $[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2]=0$ *and* $[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}]\neq 0$ *for* n *even.*

(5) $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3]=0$ *for* $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2,3$ *and* $\theta=\Sigma(\theta')\in\Sigma\pi_1(\mathbb{S}^1)$.

(6) $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_2]=0$ *for* $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2$ *and* $\theta_l=\Sigma(\theta'_l)\in\Sigma\pi_1(\mathbb{S}^1)$ *with* $l=1,2$.

(7) *If* $n\geq 1$, $\beta\in\pi_m(\mathbb{S}^{2n+1})$, $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2$, $\theta=\Sigma(\theta')\in\Sigma\pi_1(\mathbb{S}^1)$ *and* $\theta_l=\Sigma(\theta'_l)\in\Sigma\pi_1(\mathbb{S}^1)$ *with* $l=1,2,3$ *then*

(i) $[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}]\neq 0$ *for even* n *with the order* $\sharp[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}]=2$ *and* $2[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta]=0$*;*

(ii) $[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}]\neq 0$ *for* n *even and* $[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2]=0$ *for* n *odd and* $2[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2]=0$ *for any* n*;*

(iii) $[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}], i_{n,\mathbb{C}}]\neq 0$ *for* n *even. Further,* $[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3]=0$ *for odd* n *and* $2[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3]=0$ *for any* n*;*

(iv) $[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}]\neq 0$ *for* n *even with the order* $\sharp[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}]=2$. *Further,* $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2]=0$ *for* n *odd and* $2[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2]=0$ *for any* n*;*

(8) *If* $\beta\in\pi_m(\mathbb{S}^{2n+1})$, $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2$ *and* $\theta_l=\Sigma(\theta'_l)\in\Sigma\pi_1(\mathbb{S}^1)$ *with* $l=1,2,3,4$ *then*

(i) $[[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], i_{n,\mathbb{C}}\theta_4]=0$*;*

(ii) $[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], \gamma_{n,\mathbb{C}}\beta_2]=0$.

(9) *If* $\beta_k\in\pi_{m_k}(\mathbb{S}^{2n+1})$ *with* $k=1,2,3$ *and* $\theta_l=\Sigma(\theta'_l)\in\Sigma\pi_1(\mathbb{S}^1)$ *with* $l=1,2,3$ *then*

(i) $[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3]$*;*

(ii) $[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_3]=0$.

Proof. (1): It follows directly from [13, Proposition 1.7].

(2): Notice that $i_{1,\mathbb{C}}=\iota_2$ and $[\iota_2,\iota_2]=\pm 2\eta_2$.

For $n \geq 2$, we have $[i_{n,\mathbb{C}}, i_{n,\mathbb{C}}] : \mathbb{S}^3 \to \mathbb{C}P^n$. This implies $[i_{n,\mathbb{C}}, i_{n,\mathbb{C}}] = 0$. Since, $\Sigma\pi_1(\mathbb{S}^1) = \pi_2(\mathbb{S}^2)$ is the infinite cyclic group generated by ι_2, we simply derive that $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2] \neq 0$ for $n = 1$ and $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2] = 0$ for $n \geq 2$, and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_1(\mathbb{S}^1)$.

(3): Since the functor Σ annihilates the Whitehead product and $h_2([\beta_1, \beta_2]) = 0$ (Lemma 1.9(2)), we get

$$\begin{aligned}[][\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}] &= [\gamma_{n,\mathbb{C}}[\beta_1, \beta_2], i_{n,\mathbb{C}}] \\ &= \gamma_{n,\mathbb{C}}(\eta_{2n+1} \circ \Sigma([\beta_1, \beta_2]) + [\iota_{2n+1}, \eta_{2n+1}]\Sigma(h_2([\beta_1, \beta_2]))) = 0\end{aligned}$$

for any $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$. Then, Proposition 1.1(5) leads to $[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta] = 0$ for any $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$.

(4): Let $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$. Then, $[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta] = [\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}]\Sigma^{m_1}(\theta')$ implies

$$\begin{aligned}&[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2] \\ =&[\gamma_{n,\mathbb{C}}(\eta_{2n+1}\Sigma(\beta_1) + [\iota_{2n+1}, \eta_{2n+1}]\Sigma(h_2(\beta_1))), \gamma_{n,\mathbb{C}}\beta_2]\Sigma^{m_1+m_2-1}(\theta') \\ =&\gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1) + [\iota_{2n+1}, \eta_{2n+1}]\Sigma(h_2(\beta_1)), \beta_2]\Sigma^{m_1+m_2-1}(\theta') \\ =&\gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta') + \gamma_{n,\mathbb{C}}[[\iota_{2n+1}, \eta_{2n+1}]\Sigma(h_2(\beta_1)), \beta_2]\Sigma^{m_1+m_2-1}(\theta').\end{aligned}$$

Since $[[\iota_{2n+1}, \eta_{2n+1}], \iota_{2n+1}] = 0$, it follows that $\gamma_{n,\mathbb{C}}[[\iota_{2n+1}, \eta_{2n+1}]\Sigma h_2(\beta_1), \beta_2] = 0$ (Proposition 1.1(5)) so that $[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta')$. Therefore, $2[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2] = 0$ for any n and $[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2] = 0$ for odd n, since $[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta] = 0$ (Proposition 1.11(2)).

In particular, $[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}([\eta_{2n+1}, \iota_{2n+1}])$ for even n. Then, in view of [14, (1.15)], the relation $[\eta_{2n+1}, \iota_{2n+1}] \neq 0$ leads to $[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}([\eta_{2n+1}, \iota_{2n+1}]) \neq 0$ for even n.

(5): By the proof of (4), we have $[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta')$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$. Hence, by Proposition 1.1(4),

$$\begin{aligned}[][[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3] &= \gamma_{n,\mathbb{C}}[[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta'), \beta_3] \\ &= \gamma_{n,\mathbb{C}}[[\eta_{2n+1}\Sigma(\beta_1), \beta_2], \beta_3]\Sigma^{m_1+m_2+m_3-2}(\theta') = 0\end{aligned}$$

for any $\beta_3 \in \pi_{m_3}(\mathbb{S}^{2n+1})$.

(6): Let $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2$. By the proof of (4), we have $[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta'_1)$. Hence,

$$\begin{aligned}[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_2] &= [\gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2]\Sigma^{m_1+m_2-1}(\theta'_1), i_{n,\mathbb{C}}\theta_2] \\ &= [\gamma_{n,\mathbb{C}}[\eta_{2n+1}\Sigma(\beta_1), \beta_2], i_{n,\mathbb{C}}\theta_2]\Sigma^{m_1+m_2-1}(\theta'_1)\end{aligned}$$

and (3) yields $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_2] = 0$.

(7): Let $\beta \in \pi_m(\mathbb{S}^{2n+1})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$, $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2, 3$.

(i): Since the order $\sharp\eta_{2n+1} = 2$, we have $[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}(\eta_{2n+1}) \neq 0$ if and only if n is even and the rest is obvious.

(ii): Let $\beta \in \pi_m(\mathbb{S}^{2n+1})$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2$. Then, $[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2] = [[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}]\Sigma^m(\theta'_1 \wedge \theta'_2)$.
Next, $[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}] = [\gamma_{n,\mathbb{C}}(\eta_{2n+1}\Sigma(\beta) + [\iota_{2n+1}, \eta_{2n+1}]\Sigma h_2(\beta)), i_{n,\mathbb{C}}]$.

Since $\eta_{2n+1}\Sigma(\beta)$ and $[\iota_{2n+1}, \eta_{2n+1}]\Sigma h_2(\beta) = [\iota_{2n+1}, \iota_{2n+1}]\eta_{4n+1}\Sigma h_2(\beta)$ are suspensions ([14, (1.90)]), Lemma 1.9(1) yields $h_2(\eta_{2n+1}\Sigma(\beta) + [\iota_{2n+1}, \eta_{2n+1}] \circ \Sigma h_2(\beta)) = 0$. Consequently, we deduce that

$$[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2] = \gamma_{n,\mathbb{C}}(\eta^2_{2n+1}\Sigma^2(\beta))\Sigma^m(\theta'_1 \wedge \theta'_2).$$

This implies that $2[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2] = 0$ for any n and $[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2] = 0$ for odd n, since $[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta] = 0$ (Proposition 1.11(2)). In particular, $[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}(\eta^2_{2n+1}) \neq 0$ for n even.

(iii): Notice that (7)(ii) leads to $[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3] = 0$ for odd n. By the proof of (ii) for n even and Lemma 1.9(1), we get

$$\begin{aligned}[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3] &= [\gamma_{n,\mathbb{C}}(\eta^2_{2n+1}\Sigma^2(\beta))\Sigma^m(\theta'_1 \wedge \theta'_2), i_{n,\mathbb{C}}\theta_3] \\ &= [\gamma_{n,\mathbb{C}}(\eta^2_{2n+1}\Sigma^2(\beta)), i_{n,\mathbb{C}}]\Sigma^m(\theta'_1 \wedge \theta'_2 \wedge \theta'_3) \\ &= \gamma_{n,\mathbb{C}}(\eta^3_{2n+1}\Sigma^3(\beta))\Sigma^m(\theta'_1 \wedge \theta'_2 \wedge \theta'_3)\end{aligned}$$

for any $\theta_3 = \Sigma(\theta'_3) \in \Sigma\pi_1(\mathbb{S}^1)$.

This implies that $2[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3] = 0$ for n even. In particular, $[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}], i_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}(\eta^3_{2n+1}) \neq 0$ for even n.

(iv): By the proof of (ii), we have $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[\eta^2_{2n+1}\Sigma^2(\beta))\Sigma^{m_1}(\theta'_1 \wedge \theta'_2), \beta_2]$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$. Consequently, $2[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2] = 0$ for any n and $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2] = 0$ for n odd.

In particular, $[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}]\gamma_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}[\eta^2_{2n+1}, \iota_{2n+1}]$ for n even.

But, in view of [14, (1.15)], we have that $[\eta^2_{2n+1}, \iota_{2n+1}] \neq 0$ for n even. Hence, $[[[\gamma_{n,\mathbb{C}}, i_{n,\mathbb{C}}], i_{n,\mathbb{C}}], \gamma_{n,\mathbb{C}}] \neq 0$ for n even.

(8): Let $\beta \in \pi_m(\mathbb{S}^{2n+1})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2, 3, 4$.

(i): In view of the proof of (7)(iii), we have

$$[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3] = \gamma_{n,\mathbb{C}}(\eta^3_{2n+1}\Sigma^3(\beta))\Sigma^m(\theta'_1 \wedge \theta'_2 \wedge \theta'_3).$$

Hence, $\eta^4_{2n+1} = 0$ and Lemma 1.9(1) imply $[[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], i_{n,\mathbb{C}}\theta_4] = \gamma_{n,\mathbb{C}}(\eta^4_{2n+1}\Sigma^3(\beta))\Sigma^m(\theta'_1 \wedge \theta'_2 \wedge \theta'_3 \wedge \theta'_4)$ for $\theta_4 = \Sigma(\theta'_4) \in \Sigma\pi_1(\mathbb{S}^1)$.

(ii): Again, in view of the proof of (7)(iii), we have

$$[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[\eta^3_{2n+1}\Sigma^2(\beta_1)\Sigma^{m_1}(\theta'_1 \wedge \theta'_2 \wedge \theta'_3), \beta_2].$$

But, by $\eta^3_3 = 2\nu'$ and $\eta^3_{2n+1} = 4\nu_{2n+1}$ for $n \geq 2$, we have $[\eta^3_{2n+1}, \iota_{2n+1}] = 0$. Consequently, by Proposition 1.1(5), we derive that

$$[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], \gamma_{n,\mathbb{C}}\beta_2] = 0$$

for any $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2, 3$.

(9): By the proof of (7)(ii), we have $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2] = \gamma_{n,\mathbb{C}}[(\eta^2_{2n+1}\Sigma^2(\beta))\Sigma^{m_1}(\theta'_1 \wedge \theta'_2), \beta_2]$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$. This implies that

(i): in view of (1)(ii), we have

$$[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3] = [\gamma_{n,\mathbb{C}}[(\eta^2_{2n+1}\Sigma^2(\beta_1))\Sigma^{m_1}(\theta'_1 \wedge \theta'_2), \beta_2], \gamma_{n,\mathbb{C}}\beta_3] = 0$$

for $\beta_3 \in \pi_{m_3}(\mathbb{S}^{2n+1})$;

(ii): in view of (3), we have

$$[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_3] = [\gamma_{n,\mathbb{C}}[(\eta^2_{2n+1}\Sigma^2(\beta_1))\Sigma^{m_1}(\theta'_1 \wedge \theta'_2), \beta_2], i_{n,\mathbb{C}}\theta_3]$$

for $\theta_3 = \Sigma(\theta'_3) \in \Sigma\pi_1(\mathbb{S}^1)$ and the proof is complete. □

Then, the results in Lemma 1.14 imply the following

Proposition 1.15. (1) *All iterated Whitehead products in $\mathbb{C}P^n$ are annihilated by 2 provided $n > 1$. If n is odd then all triple Whitehead products vanish and there are non-trivial double*

Whitehead products on $\mathbb{C}P^n$. If n is even then there are non-trivial quadruple Whitehead products in $\mathbb{C}P^n$.

(2) *All quintuple Whitehead products in $\mathbb{C}P^n$ vanish for $n \geq 1$.*

(3) *All iterated Whitehead products in $\mathbb{C}P^n_{(p)}$ vanish for any prime $p > 2$ and $n > 1$.*

Proof. By Lemma 1.3 and Remark 1.13, it is easy to see that it suffices to show our assertions for the iterated m-fold Whitehead products where the entries are of the form $\gamma_{n,\mathbb{C}}\beta$, $i_{n,\mathbb{C}}\theta$ with $\beta \in \pi_m(\mathbb{S}^{2n+1})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_1(\mathbb{S}^1)$.

(1): Certainly, by Lemma 1.14, all Whitehead products in $\mathbb{C}P^n$ are annihilated by 2 for $n > 1$, all triple Whitehead products in $\mathbb{C}P^n$ are trivial for n odd and there are non-trivial quadruple Whitehead products in $\mathbb{C}P^n$ for n even.

Now we concentrate on the case when n is even and $n > 2$.

First, observe that just for a double Whitehead product $[-,-]$, it suffices to consider three cases: $[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2]$, $[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2]$ and $[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta]$, since the fourth case $[i_{n,\mathbb{C}}\theta, \gamma_{n,\mathbb{C}}\beta] = \pm[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta]$. So, we are left with 12 cases:

(i) $[[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3], \gamma_{n,\mathbb{C}}\beta_4]$;
(ii) $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3]$;
(iii) $[[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta], \gamma_{n,\mathbb{C}}\beta_3]$;
(iv) $[[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3], i_{n,\mathbb{C}}\theta]$;
(v) $[[[\gamma_{n,\mathbb{C}}\beta_1, \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2]$;
(vi) $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2]$;
(vii) $[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_2]$;
(viii) $[[[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_1], \gamma_{n,\mathbb{C}}\beta_2]$;
(ix) $[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_3]$;
(x) $[[[i_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], \gamma_{n,\mathbb{C}}\beta_2]$;
(xi) $[[[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta], i_{n,\mathbb{C}}\theta_3]$;
(xii) $[[[i_{n,\mathbb{C}}\theta_1, i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], i_{n,\mathbb{C}}\theta_4]$
for $\beta \in \pi_m(\mathbb{S}^{2n+1})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{2n+1})$ with $k = 1, 2, 3, 4$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_1(\mathbb{S}^1)$ with $l = 1, 2, 3, 4$.

Of the elements (i) - (xii), the following vanish:
(i) by Lemma 1.14(1)(ii);
(ii) by Lemma 1.14(5);

(iii) by Lemma 1.14(3);
(iv) by Lemma 1.14(1)(ii);
(v) by Lemma 1.14(3);
(vii) by Lemma 1.14(6);
(viii) by Lemma 1.14(2);
(ix) by Lemma 1.14(7)(ii);
(xi) by Lemma 1.14(2);
(xii) by Lemma 1.14(2).

Next, 2 annihilates (vi) and (x) by Lemma 1.14(7)(iii) and Lemma 1.14(7)(iv), respectively.

(2): By the proof of (1), the only quadruple iterated Whitehead products which is possibly non-trivial are the ones given by:
$[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2]$;
$[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3]$.

As a consequence, the only possible non-vanishing quintuple iterated Whitehead products in $\mathbb{C}P^n$ are

(i) $[[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], i_{n,\mathbb{C}}\theta_4]$;
(ii) $[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], i_{n,\mathbb{C}}\theta_3], \gamma_{n,\mathbb{C}}\beta_2]$;
(iii) $[[[[\gamma_{n,\mathbb{C}}\beta_1, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], \gamma_{n,\mathbb{C}}\beta_3]$;
(iv) $[[[[\gamma_{n,\mathbb{C}}\beta, i_{n,\mathbb{C}}\theta_1], i_{n,\mathbb{C}}\theta_2], \gamma_{n,\mathbb{C}}\beta_2], i_{n,\mathbb{C}}\theta_3]$.

The quintuple Whitehead product given in (i) vanishes by Lemma 1.14(8)(i).

The quintuple Whitehead product given in (ii) vanishes by Lemma 1.14(8)(ii).

The quintuple Whitehead product given in (iii) vanishes by Lemma 1.14(9)(i).

The quintuple Whitehead product given in (iv) vanishes by Lemma 1.14(9)(ii).

(3): By (1), all iterated Whitehead products in $\mathbb{C}P^n$ are annihilated by 2 provided $n > 1$. Hence, all iterated Whitehead products in $\mathbb{C}P^n_{(p)}$ vanish for any prime $p > 2$ and $n > 1$. This completes the proof. □

Then, in view of [23] and (1.2), Lemma 1.14 and Proposition 1.15 yield

Corollary 1.16. (1) W-long$\,(\mathbb{C}P^n) = 1$ *if and only if* $n = 3$*;*

(2) W-long $(\mathbb{C}P^n) = \begin{cases} 2 \textit{ for } n \textit{ odd unless } n = 3, \\ 4 \textit{ for } n \textit{ even}; \end{cases}$

Furthermore, if p *is a prime then*

(3) W-long $(\mathbb{C}P^1_{(p)}) = 2$*;*

(4) W-long $(\mathbb{C}P^n_{(p)}) = 1$ *for* $n \geq 2$ *and* $p \geq 3$*;*

(5) W-long $(\mathbb{C}P^3_{(p)}) = 1$*;*

(6) W-long $(\mathbb{C}P^n_{(2)}) = \begin{cases} 2 \textit{ for } n \geq 5 \textit{ odd}, \\ 4 \textit{ for } n \textit{ even}. \end{cases}$

Proof. (1): Notice that $[\gamma_{3,\mathbb{C}}\beta_1, \gamma_{3,\mathbb{C}}\beta_2] = \gamma_{3,\mathbb{C}}[\beta_1, \beta_2] = 0$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^7)$ with $k = 1, 2$. Then, by Proposition 1.11(2), we deduce that W-long $(\mathbb{C}P^3) = 1$. On the other hand, $[\gamma_{n,\mathbb{C}}, \gamma_{n,\mathbb{C}}] = \gamma_{n,\mathbb{C}}[\iota_{2n+1}, \iota_{2n+1}] \neq 0$ unless $n = 1, 3$. But for $n = 1$ we have W-long $(\mathbb{C}P^1) =$ W-long $(\mathbb{S}^2) = 2$ so (1) follows.

(2)-(6): Those are direct consequences of Lemma 1.14 and this concludes the proof. ☐

Remark 1.17. Corollary 1.16(2) improves the lower bound $3 \leq \operatorname{nil} \Omega(\mathbb{C}P^n)$ for even n given by [8, Propositions 1.4].

Now, we examine iterated Whitehead products in the space $\mathbb{H}P^n$. From equation (1.6) we have $\pi_m(\mathbb{H}P^n) = \gamma_{n,\mathbb{H}*}\pi_m(\mathbb{S}^{4n+3}) \oplus i_{n,\mathbb{H}*}\Sigma\pi_{m-1}(\mathbb{S}^3)$.

Remark 1.18. The double Whitehead products in $\mathbb{H}P^n$ are

(1) $[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2]$ $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$;

(2) $[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta]$ for $\beta \in \pi_m(\mathbb{S}^{4n+3})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$;

(3) $[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2]$ for $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ with $l = 1, 2$.

Furthermore,

(4) $[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta] = [\gamma_{n,\mathbb{H}}\beta, i_{\mathbb{H}}]\Sigma^m\theta'$ for $\beta \in \pi_m(\mathbb{S}^{4n+3})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$;

(5) $[i_{\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2] = [i_{n,\mathbb{H}}, i_{n,\mathbb{H}}]\Sigma(\theta'_1 \wedge \theta'_2)$ for $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ with $l = 1, 2$.

(6) Proposition 1.11(3) leads to the order $\sharp[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = \frac{24}{\gcd(24,n+1)}$ and $24[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}] = 0$ for $\beta \in \pi_m(\mathbb{S}^{4n+3})$.

Despite the fact that we intend to use in Section 3 only some of the results presented below, their content seems to be interesting in its own right.

Lemma 1.19. (1) *If* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *for* $k=1,2,3$ *then*

(i) $2[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2] = 2\gamma_{n,\mathbb{H}}[\beta_1,\beta_2] = 0$;

(ii) $[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3] = \gamma_{n,\mathbb{H}}[[\beta_1,\beta_2],\beta_3] = 0$.

(2) $[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] \neq 0$ *with the order* $\sharp[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = 12$ *and* $12[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2] = 0$ *for* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2$.

(3) $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] \neq 0$ *and* $3[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = 0$ *for* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2,3$.

(4) $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4] = 0$ *for* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2,3,4$.

(5) $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta] = 0$ *for any* $\beta \in \pi_m(\mathbb{S}^{4n+3})$ *and* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2,3$.

(6) $[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta] = 0$ *for* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *with* $k=1,2$ *and* $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$.

(7) *If* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *for* $k=1,2$ *and* $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$ *then* $2[[\gamma_{n,\mathbb{H}}\beta_1, i_{\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = 0$. *If* n *is odd then* $[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = 0$ *and* $[[\gamma_{n,\mathbb{H}}, i_{\mathbb{H}}], \gamma_{n,\mathbb{H}}] \neq 0$ *for* n *even.*

(8) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3] = 0$ *for* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *with* $k=1,2,3$ *and* $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$.

(9) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_2] = 0$ *for* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *with* $k=1,2$ *and* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2$.

(10) *If* $n \geq 1$, $\beta \in \pi_m(\mathbb{S}^{4n+3})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *with* $k=1,2$ *and* $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$, $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2,3$ *then:*

(i) $[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}] \neq 0$ *for* $n+1 \not\equiv 0 \,(\mathrm{mod}\ 24)$ *with the order* $\sharp[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = \frac{24}{(n+1,24)}$. *Further,* $\frac{24}{(n+1,24)}[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta] = 0$ *for* $\beta \in \pi_m(\mathbb{S}^{4n+3})$ *and* $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$;

(ii) $[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] \neq 0$ *for* n *even. Further,* $[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{\mathbb{H}}\theta_2] = 0$ *for* n *odd and* $2[[\gamma_{n,\mathbb{H}}\beta, i_{\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = 0$ *for any* n, $\beta \in \pi_m(\mathbb{S}^{4n+3})$ *and* $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2$;

(iii) $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] \neq 0$ *for* n *even. Further, for* n *odd* $[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{\mathbb{H}}\theta_3] = 0$ *and* $2[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = 0$ *for* n *arbitrary and* $\beta \in \pi_m(\mathbb{S}^{4n+3})$, $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ *with* $l=1,2,3$;

(iv) $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] \neq 0$ *if and only if* n *is even and* $n \neq 2^{t+1}-2$ *for* ≥ 1, *and the order* $\sharp[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] = 2$. *Further, for* n *either odd or* $n = 2^{l+1}-2$ *we have* $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2] = 0$ *and* $2[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2] = 0$ *for any* n, $\beta_k \in \pi_m(\mathbb{S}^{4n+3})$ *with* $k = 1, 2$ *and* $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ *with* $l = 1, 2$.

(v) $[[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4] = 0$ *for any* $\beta \in \pi_m(\mathbb{S}^{4n+3})$ *and* $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ *with* $l = 1, 2, 3, 4$;

(vi) $[[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta_2] = 0$ *for any* $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ *with* $k = 1, 2$ *and any* $\theta_l = \Sigma(\theta_l') \in \pi_{m_l'-1}(\mathbb{S}^3)$ *with* $l = 1, 2, 3$.

(11) *If* $n \geq 1$ *then* $[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta] = 0$ *for any* $\beta \in \pi_m(\mathbb{S}^{4n+3})$ *and* $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ *with* $l = 1, 2$.

Proof. (1): It follows directly from Proposition 1.1(4).

(2): Since $[\iota_4, \iota_4] = \pm(2\nu_4^+ - \Sigma\nu'^+)$ ([25, (5.8)]) with $\nu'^+ = \nu' - \alpha_1(3)$ and $i_{n,\mathbb{H}}\nu_4^+ = 0$, we conclude that $[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}[\iota_4, \iota_4] = i_{n,\mathbb{H}}(\Sigma(\nu'^+))$. Then, the order $\sharp\Sigma(\nu'^+) = 12$ implies $12[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = 0$.

Next, $12[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = [12 i_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = 0$, in view of Proposition 1.1(5), implies $[(12 i_{n,\mathbb{H}})\theta_1, i_{n,\mathbb{H}}\theta_2] = 0$ for $\theta_l \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2$. Since θ_1 is a suspension, $(12 i_{n,\mathbb{H}})\theta_1 = 12(\theta_1 i_{n,\mathbb{H}})$. Thus, $12[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2] = 0$.

(3): First, (2) leads to $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}[\Sigma(\nu'^+), \iota_4] = i_{n,\mathbb{H}}\Sigma(\nu'^+)\Sigma^4(\nu'^+)$ and, in view of [14, (1.25) and (1.28)], we have $\Sigma^2(\nu') = 2\nu_5$, $\nu' \circ \nu_6 = 0$. Then, we conclude that $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}\alpha_1(4)\alpha_1(7)$. But, by [14, (1.8)], the order $\sharp\alpha_1(4)\alpha_1(7) = 3$. Hence, we derive from the above that $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] \neq 0$ and $3[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0$.

Next, $3[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = [[3 i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0$ and Proposition 1.1(5) lead to $3[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = [[(3 i_{n,\mathbb{H}})\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = 0$ for $\theta_l \in \Sigma\pi_{m_l-1}(\mathbb{S}^3)$ with $l = 1, 2, 3$.

(4): First, notice that $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}\alpha_1(4)\alpha_1(7)$ from the proof of (3) implies $[[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}[\alpha_1(4)\alpha_1(7), \iota_4] = i_{n,\mathbb{H}}[\iota_4, \iota_4]\Sigma^4(\alpha_1(3)\alpha_1(6)) = i_{n,\mathbb{H}}[\iota_4, \iota_4]\alpha_1(7)\alpha_1(10)$. But, in view of [14, (1.8)], we have $\alpha_1(7)\alpha_1(10) = 0$. Hence, we deduce that $[[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0$.

Next, $[[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0$ and Proposition 1.1(5) yield $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4] = 0$ for any $\theta_l \in \Sigma\pi_{m_l-1}(\mathbb{S}^3)$ with $l = 1, 2, 3, 4$.

(5): First, the relation $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = i_{n,\mathbb{H}}\alpha_1(4)\alpha_1(7)$ yields $[[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta] = [i_{n,\mathbb{H}}, \gamma_{n,\mathbb{H}}\beta]\alpha_1(3+m)\alpha_1(6+m) = 0$. for any $\beta \in \pi_m(\mathbb{S}^{4n+3})$.

Next, $[[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta] = 0$ and Proposition 1.1(5) imply $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta] = 0$.

(6): Since the suspension functor Σ annihilates the Whitehead product and $h_2([\beta_1, \beta_2]) = 0$ (Lemma 1.9(2)), we get

$$\begin{aligned}[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}] &= [\gamma_{n,\mathbb{H}}[\beta_1, \beta_2], i_{n,\mathbb{H}}] \\ &= \pm(n+1)\gamma_{n,\mathbb{H}}(\nu^+_{4n+3} \circ \Sigma^3([\beta_1, \beta_2]) + [\iota_{4n+3}, \nu_{4n+3}] \circ \Sigma^3(h_2([\beta_1, \beta_2]))) = 0\end{aligned}$$

for any $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$. Then, Proposition 1.1(5) leads to $[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta] = 0$ for any $\theta \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$.

(7): Let $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$ and $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$. Then, $[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta] = [\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}]\Sigma^{m_1}(\theta')$ implies

$$\begin{aligned}&[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] \\ &\quad = [\pm(n+1)\gamma_{n,\mathbb{H}}(\nu^+_{4n+3}\Sigma^3(\beta_1) + [\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta_1)), \gamma_{n,\mathbb{H}}\beta_2]\Sigma^{m_1}(\theta') \\ &\quad = \pm(n+1)\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1) + [\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta_1)), \beta_2]\Sigma^{m_1}(\theta') \\ &\quad = \pm(n+1)\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta') \pm (n+1)\gamma_{n,\mathbb{H}}[[\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta_1)), \beta_2]\Sigma^{m_1}(\theta').\end{aligned}$$

But, $[[\iota_{4n+3}, \nu_{4n+3}], \iota_{4n+3}] = 0$ and Proposition 1.1(5) yields $\gamma_{n,\mathbb{H}}[[\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta_1)), \beta_2] = 0$. Consequently,

$$[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = \pm(n+1)\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta').$$

Therefore, $2[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = 0$ for any n and $[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = 0$ for odd n.

In particular, $[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] = \gamma_{n,\mathbb{H}}[\nu^+_{4n+3}, \iota_{4n+3}]$ for even n. Then, [14, (1.32)] leads to $[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] \neq 0$ for even n.

(8): By the proof of (7), we $[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2] = \pm(n+1)\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta')$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$ and $\theta = \Sigma(\theta') \in \Sigma\pi_{m'-1}(\mathbb{S}^3)$. Hence, by Proposition 1.1(4)-(5), we have

$$[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3] = \pm(n+1)\gamma_{n,\mathbb{H}}[[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta'), \beta_3] = 0$$

for any $\beta_3 \in \pi_{m_3}(\mathbb{S}^{4n+3})$.

(9): Let $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2$. But, by the proof of (7), we $[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], \gamma_{n,\mathbb{H}}\beta_2] = \pm(n+1)\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta_1')$. Hence,

$$\begin{aligned}
[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_2] &= \pm(n+1)[\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2]\Sigma^{m_1}(\theta_1'), i_{n,\mathbb{H}}\theta_2] \\
&= \pm(n+1)[\gamma_{n,\mathbb{H}}[\nu^+_{4n+3}\Sigma^3(\beta_1), \beta_2], i_{n,\mathbb{H}}\theta_2]\Sigma^{m_1+m_2'-1}(\theta_1')
\end{aligned}$$

and (6) yields $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_2] = 0$.

(10): Let $n \geq 1$, $\beta \in \pi_m(\mathbb{S}^{4n+3})$, $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2, 3$.

(i): Since the order $\sharp\nu^+_{4n+3} = 24$, we have $[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = \pm(n+1)\gamma_{n,\mathbb{H}}(\nu^+_{4n+3}) \neq 0$ if and only if $n+1 \not\equiv 0 \,(\mathrm{mod}\ 24)$ and the rest is obvious.

(ii): Let $\beta \in \pi_m(\mathbb{S}^{4n+3})$ and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2$. Then, $[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = [[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}]\Sigma^{m_1}(\theta_1' \wedge \theta_2')$ Next,

$$\begin{aligned}
[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] &= [\pm(n+1)\gamma_{n,\mathbb{H}}(\nu^+_{4n+3}\Sigma^3(\beta) + [\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta)), i_{n,\mathbb{H}}] \\
&= \pm(n+1)[\gamma_{n,\mathbb{H}}(\nu^+_{4n+3}\Sigma^3(\beta) + [\iota_{4n+3}, \nu_{4n+3}]\Sigma^3(h_2(\beta)), i_{n,\mathbb{H}}].
\end{aligned}$$

Since $\nu^+_{4n+3}\Sigma^3(\beta)$ and $[\iota_{4n+3}, \nu_{4n+3}] \circ \Sigma^3(h_2(\beta)) = [\iota_{4n+3}, \iota_{4n+3}]\nu_{8n+5} \circ \Sigma^3(h_2(\beta))$ ([14, (1.91)]) are suspensions, by Lemma 1.9(1), we get $h_2(\nu^+_{4n+3}\Sigma^3(\beta) + [\iota_{4n+3}, \nu_{4n+3}] \circ \Sigma^3(h_2(\beta)) = 0$. Consequently, we deduce that

$$[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = (n+1)^2\gamma_{n,\mathbb{H}}(\nu^2_{4n+3}\Sigma^6\beta)\Sigma^{m_1}(\theta_1' \wedge \theta_2').$$

This implies that $2[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = 0$ for arbitrary n and $[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = 0$ for odd n. In particular, $[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = \gamma_{n,\mathbb{H}}(\nu^2_{4n+3}) \neq 0$ for even n.

(iii): By (ii), and $h_2(\nu^2_{4n+3}\Sigma^6\beta) = 0$ (Lemma 1.9(1)), we get that

$$\begin{aligned}
[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] &= (n+1)^2[\gamma_{n,\mathbb{H}}(\nu^2_{4n+3}\Sigma^6\beta)\Sigma^m(\theta_1' \wedge \theta_2'), i_{n,\mathbb{H}}\theta_3] \\
&= (n+1)^2[\gamma_{n,\mathbb{H}}(\nu^2_{4n+3}\Sigma^6\beta), i_{n,\mathbb{H}}]\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3') \\
&= \pm(n+1)^3\gamma_{n,\mathbb{H}}(\nu^3_{4n+3}\Sigma^9\beta)\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3')
\end{aligned}$$

for any $\theta_3 = \Sigma(\theta_3') \in \Sigma\pi_{m_3'-1}(\mathbb{S}^3)$.

This implies that $2[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = 0$ for arbitrary n and
$[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3] = 0$ for odd n.

In particular, $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = \gamma_{n,\mathbb{H}}(\nu^3_{4n+3}) \neq 0$ for even n.

(iv): By (ii), we have $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}]\theta_1, i_{n,\mathbb{H}}]\theta_2, \gamma_{n,\mathbb{H}}\beta_2] = (n+1)^2\gamma_{n,\mathbb{H}}[\nu_{4n+3}^2\Sigma^6\beta)\Sigma^{m_1}(\theta_1'\wedge\theta_2'), \beta_2]$ for $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$. Consequently, $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}]\theta_1, i_{n,\mathbb{H}}]\theta_2, \gamma_{n,\mathbb{H}}\beta_2] = 0$ for n odd and $2[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}]\theta_1, i_{n,\mathbb{H}}]\theta_2, \gamma_{n,\mathbb{H}}\beta_2] = 0$ for n even.

In particular, $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] = (n+1)^2\gamma_{n,\mathbb{H}}[\nu_{4n+3}^2, \iota_{4n+3}]$.

But, in view of [14, Proposition 1.16], we deduce that $[\nu_{4n+3}^2, \iota_{4n+3}] \neq 0$ if and only if n is even unless $n = 2^{t+1}-2$ with $t \geq 1$. Hence, $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] \neq 0$ if and only if n is even unless $n = 2^{t+1} - 2$ with $t \geq 1$. Further, $[[[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}] = 0$ if and only if n is odd or $n = 2^{t+1} - 2$ with $t \geq 1$ and Proposition 1.1(5) leads to $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2] = 0$ $n = 2^{t+1} - 2$ with $t \geq 1$.

(v): In view of the proof of (iii) and $h_2(\nu_{4n+3}^3\Sigma^9(\beta)) = 0$ (Lemma 1.9(1)), we get that

$$\begin{aligned}
[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4] &= \pm(n+1)^3[\gamma_{n,\mathbb{H}}(\nu_{4n+3}^3\Sigma^9(\beta))\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3'), i_{n,\mathbb{H}}\theta_4] \\
&= \pm(n+1)^3[\gamma_{n,\mathbb{H}}(\nu_{4n+3}^3\Sigma^9(\beta)), i_{n,\mathbb{H}}]\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3' \wedge \theta_4') \\
&= (n+1)^4[\gamma_{n,\mathbb{H}}(\nu_{4n+3}^4\Sigma^{12}(\beta))\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3' \wedge \theta_4')
\end{aligned}$$

for any $\theta_4 = \Sigma(\theta_4') \in \Sigma\pi_{m_4'-1}(\mathbb{S}^3)$.

Then, $\nu_{4n+3}^4 = 0$ leads to $[[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4] = 0$.

(vi): Again, in view of the proof of (iii), we get that $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta_2] = \pm(n+1)^3\gamma_{n,\mathbb{H}}[\nu_{4n+3}^3\Sigma^9(\beta_1)\Sigma^m(\theta_1' \wedge \theta_2' \wedge \theta_3'), \beta_2]$. But, by means of [14, Proposition 1.46], we have $[\nu_{4n+3}^3, \iota_{4n+3}] = 0$. Consequently, by Proposition 1.1(5), we derive that

$$[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta_2] = 0$$

for any $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2$ and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2$.

(11) Consider the Jacobi identity (Proposition 1.1(6))

$$[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta]+[[i_{n,\mathbb{H}}, \gamma_{n,\mathbb{H}}\beta], i_{n,\mathbb{H}}]+[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = [[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta]+2[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0.$$

But, by (10)(ii), we $[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2] = (n+1)^2\gamma_{n,\mathbb{H}}(\nu_{4n+3}^2\Sigma^6\beta)\Sigma^{m_1}(\theta_1' \wedge \theta_2')$ for $\beta \in \pi_m(\mathbb{S}^{4n+3})$ and $\theta_l = \Sigma(\theta_l') \in \Sigma\pi_{m_l'-1}(\mathbb{S}^3)$ with $l = 1, 2$. Hence, we get that $2[[\gamma_{n,\mathbb{H}}\beta, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] = 0$ and so $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta] = 0$. Since, $[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta] = [[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}]\circ\Sigma(\theta_1'\wedge\theta_2'), \gamma_{n,\mathbb{H}}\beta] = [[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], \gamma_{n,\mathbb{H}}\beta]\Sigma^m(\theta_1' \wedge \theta_2')$, we deduce that

$$[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta] = 0$$

and the proof is complete. □

We mimic the proof of Proposition 1.15 to show that results presented in Lemma 1.19 imply

Proposition 1.20. (1) *All quadruple Whitehead products in $\mathbb{H}P^n$ are annihilated by 2. If n is odd then all quadruple Whitehead products vanish and there are non-trivial trivial triple Whitehead products on $\mathbb{H}P^n$. If n is even then there are non-trivial quadruple Whitehead products in $\mathbb{H}P^n$.*

(2) *All quintuple Whitehead products in $\mathbb{H}P^n$ vanish.*

(3) *All quadruple Whitehead products in $\mathbb{H}P^n_{(3)}$ vanish and there are non-trivial triple Whitehead products in $\mathbb{H}P^n_{(3)}$.*

(4) *All iterated Whitehead products in $\mathbb{H}P^n_{(p)}$ vanish for any prime $p > 3$ and $n > 1$.*

Proof. We proceed in a similar way as in the proof of Proposition 1.15 that it suffices to show our assertions for iterated m-fold Whitehead products where the entries are of the form $\gamma_{n,\mathbb{H}}\beta$, $i_{n,\mathbb{H}}\theta$ with $\beta \in \pi_m(\mathbb{S}^{4n+3})$ and $\theta = \Sigma(\theta') \in \Sigma\pi_l(\mathbb{S}^3)$.

(1): Certainly, by Lemma 1.19, all quadruple Whitehead products in $\mathbb{H}P^n$ are trivial for n odd and there are non-trivial quadruple Whitehead products in $\mathbb{H}P^n$ for n even. First, observe that just for a double Whitehead product $[-,-]$, it suffices to consider three cases: $[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2]$, $[i_{\mathbb{H}}\theta_1, i_{\mathbb{H}}\theta_2]$ and $[\gamma_{n,\mathbb{H}}\beta, i_{\mathbb{H}}\theta]$, since the fourth case is $[i_{\mathbb{H}}\theta, \gamma_{n,\mathbb{H}}\beta] = \pm[\gamma_{n,\mathbb{H}}\beta, i_{\mathbb{H}}\theta]$. Thus, we are left with 12 cases:

(i) $[[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3], \gamma_{n,\mathbb{H}}\beta_4]$;
(ii) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3]$;
(iii) $[[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta], \gamma_{n,\mathbb{H}}\beta_3]$;
(iv) $[[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3], i_{n,\mathbb{H}}\theta]$;
(v) $[[[\gamma_{n,\mathbb{H}}\beta_1, \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2]$;
(vi) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2]$;
(vii) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_2]$;
(viii) $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_1], \gamma_{n,\mathbb{H}}\beta_2]$;
(ix) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_3]$;
(x) $[[[i_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta_2]$;
(xi) $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta], i_{n,\mathbb{H}}\theta_3]$;
(xii) $[[[i_{n,\mathbb{H}}\theta_1, i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4]$
for $\beta_k \in \pi_{m_k}(\mathbb{S}^{4n+3})$ with $k = 1, 2, 3, 4$ and $\theta_l = \Sigma(\theta'_l) \in \Sigma\pi_{m'_l-1}(\mathbb{S}^3)$ with $l = 1, 2, 3, 4$.

Of the elements (i) - (xii), the following vanish:
(i) by Lemma 1.19(1)(ii);
(ii) by Lemma 1.19(8);
(iii) by Lemma 1.19(6);
(iv) by Lemma 1.19(1)(ii);
(v) by Lemma 1.19(6);
(vii) by Lemma 1.19(9);
(viii) by Lemma 1.19(11),
(x) by Lemma 1.19(5);
(xi) by Lemma 1.19(11);
(xii) by Lemma 1.19(4).

Next, 2 annihilates the elements in (vi) and (ix) by Lemma 1.19(10)(iii) and Lemma 1.19(10)(ii), respectively.

(2): By the proof of (1), the only quadruple iterated Whitehead products which are possibly non-trivial are the ones given by:
$[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2]$, and $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3]$.

As a consequence, the only possible non-vanishing quintuple iterated Whitehead product in $\mathbb{H}P^n$ are

(i) $[[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2], i_{n,\mathbb{H}}\theta_3]$;
(ii) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], \gamma_{n,\mathbb{H}}\beta_2], \gamma_{n,\mathbb{H}}\beta_3]$;
(iii) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_2], i_{n,\mathbb{H}}\theta_3], i_{n,\mathbb{H}}\theta_4]$;
(iv) $[[[\gamma_{n,\mathbb{H}}\beta_1, i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_1], i_{n,\mathbb{H}}\theta_3], \gamma_{n,\mathbb{H}}\beta_2]$.

The quintuple Whitehead product given in (i) vanishes by Lemma 1.19(10)(v).

The quintuple Whitehead product given in (ii) vanishes by Lemma 1.19(10)(iv).

The quintuple Whitehead products given in (iii) and (iv) vanish by Lemma 1.19(10)(v)-(vi), respectively.

(3): By means of Lemma 1.19(3), we have $[[i_{n,\mathbb{H}}, i_{n,\mathbb{H}}], i_{n,\mathbb{H}}] \neq 0$ on $\mathbb{H}P^n_{(3)}$ and the rest is a direct consequence of (2).

(4): In view of Remark 1.18(6), 24 annihilates any iterated Whitehead product in $\mathbb{H}P^n$ for $n > 1$. Hence, all iterated Whitehead products in $\mathbb{H}P^n_{(p)}$ vanish for any prime $p > 3$ and $n > 1$. This concludes the proof. □

Then, in view of (1.2), Lemma 1.19 and Proposition 1.20 yield

Corollary 1.21. *If $n \geq 1$ then*

(1) $\text{W-long}\,(\mathbb{H}P^n) = \begin{cases} 3 \text{ provided } n \text{ is odd,} \\ 4 \text{ provided } n \text{ is even;} \end{cases}$

(2) $\text{W-long}\,(\mathbb{H}P^n_{(p)}) = 1$ *for any prime $p > 3$;*

(3) $\text{W-long}\,(\mathbb{H}P^n_{(3)}) = 3$*;*

(5) $\text{W-long}\,(\mathbb{H}P^n_{(2)}) = \begin{cases} 3 \text{ provided } n \text{ is odd,} \\ 4 \text{ provided } n \text{ is even.} \end{cases}$

Remark 1.22. Corollary 1.21(1) improves, for even n, the lower bound $3 \leq \operatorname{nil}\Omega(\mathbb{H}P^n)$ given by [8, Propositions 1.5].

Furthermore, Corollaries 1.12, 1.16 and 1.21 together with [4, Theorem 4.6] and [23, Propositions 2.11, 2.12] yield the following result which improves [8, Propositions 1.4, 1.5] and [23, Corollary 2.13].

Theorem 1.23. (1) $\text{W-long}\,(\mathbb{R}P^n) = \operatorname{nil}\Omega(\mathbb{R}P^n) = \begin{cases} 2 \text{ for } n \text{ odd unless } n = 0, 1, 3, \\ 1 \text{ if and only if } n = 0, 1, 3; \end{cases}$

(2) $\operatorname{nil}\Omega(\mathbb{R}P^n) = \infty$ *for even n;*

(3) $\text{W-long}\,(\mathbb{C}P^n) = \operatorname{nil}\Omega(\mathbb{C}P^n) = \begin{cases} 2 \text{ for } n \text{ odd unless } n = 3, \\ 1 \text{ if and only if } n = 3; \end{cases}$

(4) $\text{W-long}\,(\mathbb{C}P^n) = 4 \leq \operatorname{nil}\Omega(\mathbb{C}P^n) \leq 6$ *for even n;*

(5) $\text{W-long}\,(\mathbb{H}P^n) = 3 \leq \operatorname{nil}\Omega(\mathbb{H}P^n)$ *for odd n;*

(6) $\text{W-long}\,(\mathbb{H}P^n) = 4 \leq \operatorname{nil}\Omega(\mathbb{H}P^n)$ *for even n;*

(7) $\text{W-long}\,(\mathbb{H}P^n) = \operatorname{nil}\Omega(\mathbb{H}P^n) = 3$ *if $n \equiv -1 \pmod{24}$.*

Remark 1.24. In view of [23, Propositions 2.11, 2.12], we have $\bar{c}_{7,\Omega(\mathbb{C}P^{2n})} = 0$. Hence, $\operatorname{nil}\Omega(\mathbb{C}P^{2n}) \leq 6$. But, in [23, Corollary 2.13] it is stated $\operatorname{nil}\Omega(\mathbb{C}P^{2n}) \leq 7$.

It is easy to see that $\operatorname{nil}\Omega(\mathbb{R}P^n_{(2)}) = 2$ unless $n = 1, 3, 7$ and $\operatorname{nil}\Omega(\mathbb{R}P^n_{(p)}) = 1$ for any prime $p > 2$ and odd n.

Then, Corollaries 1.12, 1.16 and 1.21 lead to the following completions of Meier's results [20, Corollary 4.3 and Theorem 5.4] on the homotopy nilpotency of p-localized projective spaces $\mathbb{K}P^n$.

Theorem 1.25. *Let p be a prime and $n \geq 1$. Then:*

(1) $\text{W-long}\,(\mathbb{R}P^n_{(p)}) = \operatorname{nil}\Omega(\mathbb{R}P^n_{(p)}) = \begin{cases} 2 \text{ for } p = 2 \text{ and odd } n \text{ unless } n = 1, 3, 7, \\ 1 \text{ for } p = 2 \text{ and } n = 1, 3, 7 \text{ or } p > 2 \text{ and odd } n; \end{cases}$

(2) $\text{W-long}\,(\mathbb{C}P^n_{(p)}) = \operatorname{nil}\Omega(\mathbb{C}P^n_{(p)}) = 1$ *for* $p > 2$*;*

(3) $\text{W-long}\,(\mathbb{C}P^n_{(2)}) = \operatorname{nil}\Omega(\mathbb{C}P^n_{(2)})$ *for odd* n*;*

(4) $\text{W-long}\,(\mathbb{C}P^n_{(2)}) = 4 \leq \operatorname{nil}\Omega(\mathbb{C}P^n_{(2)}) \leq 6$ *for even* n*;*

(5) $\text{W-long}\,(\mathbb{H}P^n_{(p)}) = \operatorname{nil}\Omega(\mathbb{H}P^n_{(p)}) = 1$ *for* $p > 3$ *and* $n \geq 1$*;*

(6) $\text{W-long}\,(\mathbb{H}P^n_{(3)}) = 3 \leq \operatorname{nil}\Omega(\mathbb{H}P^n_{(3)}) \leq 4$ *for* $n \geq 1$*;*

(7) $\text{W-long}\,(\mathbb{H}P^n_{(2)}) = 3 \leq \operatorname{nil}\Omega(\mathbb{H}P^n_{(2)})$ *for odd* n*;*

(8) $\text{W-long}\,(\mathbb{H}P^n_{(2)}) = 4 \leq \operatorname{nil}\Omega(\mathbb{H}P^n_{(2)})$ *for even* n*;*

(9) $\text{W-long}\,(\mathbb{H}P^n_{(3)}) = \operatorname{nil}\Omega(\mathbb{H}P^n_{(3)}) = 3$ *if* $n \equiv 2 \,(\mathrm{mod}\, 3)$*;*

(10) $\text{W-long}\,(\mathbb{H}P^n_{(3)}) = 3 \leq \operatorname{nil}\Omega(\mathbb{H}P^n_{(3)}) \leq 4$ *if* $n \geq 1$*;*

(11) $\text{W-long}\,(\mathbb{H}P^\infty_{(p)}) = \operatorname{nil}\Omega(\mathbb{H}P^\infty_{(p)}) = \operatorname{nil}\mathbb{S}^3_{(p)} =$
$\begin{cases} 2 \text{ for } p = 2 \text{ with respect to four loop multiplications on } \mathbb{S}^3_{(2)}, \\ 3 \text{ for } p = 3 \text{ with respect to two loop multiplications on } \mathbb{S}^3_{(3)}, \\ 1 \text{ for } p > 3 \text{ with respect to all multiplications on } \mathbb{S}^3_{(p)}; \end{cases}$

(12) $\text{W-long}\,(\mathbb{K}P^n_{(0)}) = \operatorname{nil}\Omega(\mathbb{K}P^n_{(0)}) = \begin{cases} 1 \text{ for odd } n \text{ and } \mathbb{K} = \mathbb{R}, \\ 1 \text{ for } n \geq 1 \text{ and } \mathbb{K} = \mathbb{C}, \mathbb{H}. \end{cases}$

Now, let $\mathcal{S}^{2n+1}$ be an odd homotopy $(2n+1)$-sphere with a free action $G \times \mathcal{S}^{2n+1} \to \mathcal{S}^{2n+1}$ of a finite group G and write $\mathcal{S}^{2n+1}/G$ for the associated quotient space. Since the fundamental group $\pi_1(\mathcal{S}^{2n+1}/G) \approx G$ acts trivially on the infinite cyclic group $\pi_n(\mathcal{S}^{2n+1}/G) \approx \mathbb{Z}$, *mutatis mutandi* the proof of [8, Theorem 1.1], we deduce

Proposition 1.26. *If a finite group G acts freely on an odd homotopy sphere $\mathcal{S}^{2n+1}$ then there is an H-homotopy equivalence*

$$\Omega(\mathcal{S}^{2n+1}/G) \simeq G \times \Omega(\mathcal{S}^{2n+1}).$$

Hence,

$$\text{W-long}\,(\mathcal{S}^{2n+1}/G) \leq 2 \leq \operatorname{nil}\Omega(\mathcal{S}^{2n+1}/G) = \max\{\operatorname{nil} G, \operatorname{nil}\Omega(\mathbb{S}^{2n+1})\}.$$

This implies that

$$\operatorname{nil}\Omega(\mathcal{S}^{2n+1}/G) < \infty$$

if and only if G is a nilpotent group.

2. Nilpotency of $[J(X), \Omega(Y)]$

Let $J(X)$ be the James construction for a space X. Cohen and Wu [6] have asked

Question 2.1. Is the Cohen group

$$[J(X), \Omega(Y)]$$

nilpotent for any spaces X, Y?

By the group isomorphism $[J(\mathbb{S}^r), \Omega(Y)] \approx [\Sigma J(\mathbb{S}^r), Y]$, it follows from $\Sigma J(\mathbb{S}^r) \simeq \bigvee_{i=1}^{\infty} \mathbb{S}^{ir+1}$ that $[J(\mathbb{S}^r), \Omega(Y)]$ is a one-to-one correspondence with the product $\prod \pi_{ir+1}(Y)$. Thus, a typical element of $[J(\mathbb{S}^r), \Omega(Y)]$ is an infinite tuple $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots)$, where $\alpha_i \in \pi_{ir+1}(Y)$ for $i \geq 1$. Then, [12] yields the group multiplication $\circledast$ on $[J(\mathbb{S}^r), \Omega(Y)]$ given by

$$(\bar{\alpha} \circledast \bar{\beta})_j = \alpha_j + \beta_j + \sum_{s+t=j} \Phi_{s,s+t-1}[\alpha_s, \beta_t]$$

for some integral coefficients $\Phi_{s,s+t-1}$.

For any one of the spheres $Y = \mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$ the group $[J(\mathbb{S}^1), \Omega(\mathbb{S}^n)]$ is Abelian, so $\operatorname{nil}([J(\mathbb{S}^1), \Omega(\mathbb{S}^n)]) = 1$ for $n = 1, 3, 7$. In view of Proposition 1.4, for an odd sphere $\mathbb{S}^n$ not one of these three, either we have $\operatorname{nil}[J(\mathbb{S}^1), \Omega(\mathbb{S}^n)]$ equals 1 or 2, which is equivalent to saying that $[J(\mathbb{S}^1), \Omega(\mathbb{S}^n)]$ is Abelian or not, respectively. Note also that the group $[J(\mathbb{S}^1), \Omega(\mathbb{S}^2)]$ was shown to be Abelian in [10].

For certain spheres, we have sufficient conditions under which $[J(\mathbb{S}^1), \Omega(\mathbb{S}^n)]$ is non-Abelian, i.e., $\operatorname{nil}[J(\mathbb{S}^1), \Omega(\mathbb{S}^n)] \geq 2$.

Let $T^*(01)$ denote the set of natural numbers n with $(n)_3 = (n_i)$ and $n_i \in \{0,1\}$ for $i \geq 1$, where $(n)_3$ is the base 3 expansion of n. Further, we write $T^*(01)-1 = \{n-1;\ n \in T^*(01)\}$.

For spheres, we have the following

Theorem 2.2. *For even spheres $\mathbb{S}^{2n}$, the group $[J(\mathbb{S}^1), \Omega(\mathbb{S}^{2n})]$ is non-Abelian if one of the following conditions is satisfied*

(1) $n = 2^\ell$ *for any* $\ell \geq 1$*;*
(2) n *is not congruent to* $0 \pmod 3$ *and* $n \in T^*(01) - 1$*;*
(3) *if* n *is odd and* $n = 2^a - 1$ *or* $n = 2^a + 2^b - 1$ *for some* $b > a \geq 0$.

For odd spheres $\mathbb{S}^{2n+1}$, suppose $n \geq 4$ and $n+1 \not\equiv 0 \pmod 8$. The group $[J(\mathbb{S}^1), \Omega\mathbb{S}^{2n+1}]$ is non-Abelian if one of the following conditions is satisfied

(1) $n = 2^\ell - 3$ *for any* $\ell \geq 0$ *when* n *is odd;*
(2) $n = 2^\ell - 2$ *for any* $\ell \geq 0$ *when* n *is even.*

Proof. First, consider the even spheres $\mathbb{S}^{2n}$. The results follow from [11, Propositions 5.8 and 5.9]. For odd spheres, we consider the Whitehead product $[\iota_{2n+1}, \sigma_{2n+1}]$. It follows from [14, p. 16] that the order $\sharp[\iota_{2n+1}, \sigma_{2n+1}] = 2$ provided $n \geq 4$ and $n+1 \not\equiv 0 \pmod 8$. Following the group structure on $[J(\mathbb{S}^1), \Omega(\mathbb{S}^{2n+1})]$ from [12], the product $\iota_{2n+1} \circledast \sigma_{2n+1}$ contains the Whitehead product $[\iota_{2n+1}, \sigma_{2n+1}]$ whose coefficient is given by $\Phi_{2n,4n+6} = \binom{2n+3}{n}$. On the other hand, the product $\sigma_{2n+1} \circledast \iota_{2n+1}$ contains the Whitehead product $[\sigma_{2n+1}, \iota_{2n+1}]$ whose coefficient is given by $\Phi_{2n+7,4n+6} = 0$. Thus, the group $[J(\mathbb{S}^1), \Omega(\mathbb{S}^{2n+1})]$ is non-Abelian if $\binom{2n+3}{n}$ is odd since the order $\sharp[\iota_{2n+1}, \sigma_{2n+1}] = 2$. Next, we show that $\binom{2n+3}{n}$ is odd if and only if $n = 2^i - 2$ when n is even and $n = 2^i - 3$ when n is odd.

Using Lucas' Theorem, one can show that $\binom{2k-1}{k}$ is odd if and only if $k = 2^i$. It is easy to verify that $(n+3) \cdot \binom{2n+3}{n} = (n+1) \cdot \binom{2n+3}{n+2}$. Thus, when n is even, $\binom{2n+3}{n}$ is odd if and only if $\binom{2n+3}{n+2}$ is odd. Now $\binom{2n+3}{n+2} = \binom{2k-1}{k}$ for $k = n+2$. Hence, $\binom{2n+3}{n}$ is odd if and only if $n+2 = k = 2^i$ or $n = 2^i - 2$.

Next, suppose n is odd. It is easy to see that $(2n+5) \cdot \binom{2n+3}{n} = \frac{(n+1)}{2} \cdot \binom{2n+5}{n+3}$. If $\binom{2n+3}{n}$ is odd then $\binom{2n+5}{n+3} = \binom{2k-1}{k}$ is odd where $k = n+3$. Thus, $\binom{2n+3}{n}$ is odd implies that $n = 2^i - 3$.

Conversely, suppose $n = 2^i - 3$. It follows that $\frac{(n+1)}{2} = 2^{i-1} - 1$ is also odd since $n = 2^i - 3 \geq 1$ so that $i \geq 2$. It follows that $\binom{2n+3}{n}$ is also odd and the proof is complete. □

For odd spheres, all triple Whitehead products vanish (see [13, Proposition 1.8]) so the next result follows immediately from Theorem 2.2.

Corollary 2.3. *Given an odd sphere $\mathbb{S}^{2n+1}$, if n satisfies the conditions as in* Theorem 2.2 *then* $\operatorname{nil}[J(\mathbb{S}^1), \Omega(\mathbb{S}^{2n+1})] = 2$.

Certainly, Proposition 1.4 yields

$$\operatorname{nil}[J(X), \Omega(\mathbb{S}^n)] \leq \begin{cases} 2 \text{ for } n \text{ odd or } n = 2, \\ 3 \text{ for } n \text{ even with } n \neq 2. \end{cases}$$

Next, notice that for a path-connected space X, the space $J(X)$ has the weak homotopy type of $\Omega\Sigma(X)$ and so $\Sigma J(X)$ is 1-connected. It follows that a covering map $\tilde{Y} \to Y$ induces a group isomorphism $[J(X), \Omega(\tilde{Y})] \approx [J(X), \Omega(Y)]$ (cf. also [13, Proposition 3.1(2)]). In particular, for $Y = \mathbb{R}P^n$, $\mathcal{S}^{2n+1}/G$, we get

$$[J(X), \Omega(\mathbb{R}P^n)] = [\Sigma J(X), \mathbb{R}P^n] = [\Sigma J(X), \mathbb{S}^n] = [J(X), \Omega(\mathbb{S}^n)]$$

and

$$[J(X), \Omega(\mathcal{S}^{2n+1}/G)] = [\Sigma J(X), \mathcal{S}^{2n+1}/G] = [\Sigma J(X), \mathcal{S}^{2n+1}] = [J(X), \Omega(\mathbb{S}^{2n+1})].$$

Consequently, the canonical H-maps $\Omega(\mathbb{S}^n) \to \Omega(\mathbb{R}P^n)$ and $\Omega(\mathcal{S}^{2n+1}) \to \Omega(\mathcal{S}^{2n+1}/G)$ lead to isomorphisms

$$[J(X), \Omega(\mathbb{S}^n)] \approx [J(X), \Omega(\mathbb{R}P^n)] \text{ and } [J(X), \Omega(\mathcal{S}^{2n+1})] \approx [J(X), \Omega(\mathcal{S}^{2n+1}/G)].$$

This implies that

$$\operatorname{nil}[J(X), \Omega(\mathbb{R}P^{2n})] = \operatorname{nil}[J(X), \Omega(\mathbb{S}^{2n})] \leq 3$$

and

$$\operatorname{nil}[J(X), \Omega(\mathcal{S}^{2n+1}/G)] = \operatorname{nil}[J(X), \Omega(\mathbb{S}^{2n+1})] \leq 2$$

for $n \geq 0$ provided X is path-connected.

The above and Theorem 1.23 lead to

Corollary 2.4. *If X is any path-connected space and $n \geq 0$ then*

(1) $\operatorname{nil}[J(X), \Omega(\mathbb{R}P^{2n+1})] \leq 2$;

(2) $\operatorname{nil}[J(X), \Omega(\mathbb{C}P^{2n+1})] \leq 2$;

(3) $\operatorname{nil}[J(X), \Omega(\mathbb{C}P^{2n})] \leq 6$ *for $n \geq 1$;*

(4) $\mathrm{nil}\,[J(X), \Omega(\mathbb{H}P^n)] \leq 3$ *if* $n \equiv -1 \,(\mathrm{mod}\, 24)$*;*

(5) $\mathrm{nil}\,[J(X), \Omega(\mathbb{R}P^{2n})] = \mathrm{nil}\,[J(X), \Omega(\mathbb{S}^{2n})] \leq 3$*;*

(6) $\mathrm{nil}\,[J(X), \Omega(\mathcal{S}^{2n+1}/G)] = \mathrm{nil}\,[J(X), \Omega(\mathbb{S}^{2n+1})] \leq 2$.

For 1-connected spaces X, we use the following result in order to compare $[J(X), \Omega(\mathbb{C}P^n)]$ with $[J(X), \Omega(\mathbb{S}^{2n+1})]$.

Lemma 2.5. ([9, Lemma 3.4]) *Let* $f\colon X \to \mathbb{C}P^n$*, for* $n \geq 1$*. Then, there is a map* $\tilde{f}\colon X \to \mathbb{S}^{2n+1}$ *with* $f = \gamma_{n,\mathbb{C}}\tilde{f}$ *if and only if the induced map* $f^*\colon H^2(\mathbb{C}P^n;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ *is trivial. Equivalently, the image of the induced map* $(\gamma_{n,\mathbb{C}})_*\colon [X, \mathbb{S}^{2n+1}] \to [X, \mathbb{C}P^n]$ *is given by maps* $f\colon X \to \mathbb{C}P^n$ *such that* $f^*\colon H^2(\mathbb{C}P^n;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ *is trivial.*

In particular, the induced map $(\gamma_{n,\mathbb{C}})_*\colon [X, \mathbb{S}^{2n+1}] \to [X, \mathbb{C}P^n]$ *is a bijection provided* $[X, \mathbb{S}^1] = H^1(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = 0$.

Now, Lemma 2.5 implies

Proposition 2.6. *If* X *is a* 1*-connected space then*

$$\mathrm{nil}\,[J(X), \Omega(\mathbb{C}P^n)] \leq 2.$$

Proof. If X is a 1-connected space then $J(X) \simeq \Omega(\Sigma X)$ is 1-connected as well. This implies that $H^2(\Sigma\Omega(\Sigma X);\mathbb{Z}) = 0$ and, in view of Lemma 2.5, we derive that the canonical H-map $\Omega(\gamma_{n,\mathbb{H}}) : \Omega(\mathbb{S}^{2n+1}) \to \Omega(\mathbb{C}P^n)$ leads to a group epimorphism

$$[J(X), \Omega(\mathbb{S}^{2n+1})] \longrightarrow [J(X), \Omega(\mathbb{C}P^n)].$$

Consequently,

$$\mathrm{nil}\,[J(X), \Omega(\mathbb{C}P^n)] \leq \mathrm{nil}\,[J(X), \Omega(\mathbb{S}^{2n+1})] \leq 2$$

and the proof follows. □

As a consequence of Theorem 1.25, we have

Corollary 2.7. *Let* p *be a prime and* $n \geq 1$ *be a natural number. Then, for any path-connected space* X*,*

(1) $\mathrm{nil}\,[J(X), \Omega(\mathbb{R}P^n_{(2)})] \leq 2$ *for odd* n *unless* $n = 1, 3, 7$*;*

(2) $\mathrm{nil}\,[J(X), \Omega(\mathbb{R}P^n_{(p)})] = 1$ *for* $p > 2$ *and odd* n*;*

(3) $\operatorname{nil}[J(X),\Omega(\mathbb{C}P^n_{(p)})]=1$ *for* $p>2$ *and* $n\geq 1$*;*

(4) $\operatorname{nil}[J(X),\Omega(\mathbb{C}P^n_{(2)})]\leq \begin{cases}2 \textit{ for odd } n,\\ 6 \textit{ for even } n;\end{cases}$

(5) $\operatorname{nil}[J(X),\Omega(\mathbb{C}P^n_{(p)})]=1$ *for* $p>2$ *and* $n\geq 1$*;*

(6) $\operatorname{nil}[J(X),\Omega(\mathbb{H}P^n_{(p)})]=1$ *for* $p>3$ *and* $n\geq 1$*;*

(7) $\operatorname{nil}[J(X),\Omega(\mathbb{H}P^n_{(3)})]\leq 4$ *for* $n\geq 1$*;*

(8) $\operatorname{nil}[J(X),\Omega(\mathbb{H}P^n_{(3)})]=3$ *for* $n\equiv 2\,(\operatorname{mod} 3)$*;*

(9) $\operatorname{nil}[J(X),\Omega(\mathbb{R}P^n_{(0)})]=1$ *for odd* n *and* $\operatorname{nil}[J(X),\Omega(\mathbb{K}P^n_{(0)})]=1$ *for* $\mathbb{K}=\mathbb{C},\,\mathbb{H}$ *and* $n\geq 1$.

3. Exponent of $[\Omega(\mathbb{S}^{r+1}),\Omega(Y)]$

In [13], we studied the exponents of the total Cohen groups $[\Omega(\mathbb{S}^{r+1}),\Omega(Y)]$ where Y is a sphere, a homotopy spherical space form, or a projective space over $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$. Using the computation of the Whitehead products in $\mathbb{H}P^k$ from Section 1, we improve [13, Theorem 3.8 and Theorem 3.9].

First, we follow [13, Section 2] to examine the coordinates of the powers $\bar{\alpha}^M$ with $M\geq 1$ for $\bar{\alpha}\in[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{H}P^n)]$ with $n\leq\infty$.

Suppose $\alpha_k\in\pi_{kr+1}(\mathbb{H}P^n),\alpha_l\in\pi_{lr+1}(\mathbb{H}P^n)$ with $1\leq k<l$. Denote by (α_k,α_l) the element $(0,\ldots,0,\alpha_k,0,\ldots,0,\alpha_l,0,\ldots)\in[\Omega(\mathbb{S}^{r+1}),\Omega(\mathbb{H}P^n)]$, the infinite sequence with entries α_k, α_l in the coordinates in positions k, l and zero otherwise. Then,

$$(\alpha_k,\alpha_l)^2=(2\alpha_l,2\alpha_l,\Phi_{k,2k-1}[\alpha_k,\alpha_k],\Phi_{l,2l-1}[\alpha_l,\alpha_l],\Phi_\Delta[\alpha_k,\alpha_l]).$$

Here,

$$\Phi_{l,k}=\begin{cases}-\binom{\frac{k}{2}}{\frac{l}{2}}, & \text{if } l \text{ is even and } k \text{ is even;}\\ 0, & \text{if } l \text{ is odd and } k \text{ is even;}\\ \binom{\frac{k-1}{2}}{\frac{l-1}{2}}, & \text{if } l \text{ is odd and } k \text{ is odd;}\\ -\binom{\frac{k-1}{2}}{\frac{l}{2}}, & \text{if } l \text{ is even and } k \text{ is odd}\end{cases}$$

for any positive integers l, k with $1 \leq l \leq k$ and

$$\Phi_\Delta = \Phi_{k,k+l-1} + (-1)^{(k+1)(l+1)}\Phi_{l,k+l-1}.$$

The possible non-zero coordinates are divided into four types: (homogenous) elements, double, triple and quadruple Whitehead products. This is the case, because all quadruple Whitehead products in the space $\mathbb{H}P^n$ for odd $n < \infty$ and in $\mathbb{H}P^n_{(3)}$ for $n < \infty$ vanish (Proposition 1.20) and for $n = \infty$ ([13, Proposition 3.7]). Further, all quintuple Whitehead products in $\mathbb{H}P^n_{(2)}$ vanish for even $n < \infty$ (Proposition 1.20).

Now, we compute $(\alpha_k, \alpha_l)^3$. It is easy to see that the first type consists of $3\alpha_k$ and $3\alpha_l$. Second type coordinates are: $3\Phi_{k,2k-1}[\alpha_k, \alpha_k]$, $3\Phi_{l,2l-1}[\alpha_l, \alpha_l]$, and $3\Phi_\Delta[\alpha_k, \alpha_l]$. Finally, for the triple Whitehead products, we have

$$\Phi_{k,2k-1}\Phi_{2k,3k-1}[[\alpha_k, \alpha_k], \alpha_k], \Phi_{k,2k-1}\Phi_{2k,2k+l-1}[[\alpha_k, \alpha_k], \alpha_l],$$

$$\Phi_{l,2l-1}\Phi_{2l,2l+k-1}[[\alpha_l, \alpha_l], \alpha_k], \Phi_{l,2l-1}\Phi_{2l,2l+k-1}[[\alpha_l, \alpha_l], \alpha_l],$$

$$\Phi_\Delta\Phi_{k+l,2k+l-1}[[\alpha_k, \alpha_l], \alpha_k], \Phi_\Delta\Phi_{k+l,2l+k-1}[[\alpha_k, \alpha_l], \alpha_l].$$

To continue in this fashion, the power $(\alpha_k, \alpha_l)^M$ has the following possibly non-zero coordinates:

Type I: $M\alpha_k, M\alpha_l$;

Type II: $(1 + 2 + 3 + \ldots + (M-1))\Phi_{k,2k-1}[\alpha_k, \alpha_k] = \binom{M}{2}\Phi_{k,2k-1}[\alpha_k, \alpha_k]$, $\binom{M}{2}\Phi_{l,2l-1}[\alpha_l, \alpha_l]$, $\binom{M}{2}\Phi_\Delta[\alpha_k, \alpha_l]$;

Type III: each of triple Whitehead products will have the following as a factor in its coefficient:

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \cdots + (1+2+\cdots+M-1).$$

This sum in turn is equal to

$$\sum_{m=2}^{M-1}\binom{m}{2}.$$

As it turns out [11, p. 120], we have

$$\sum_{m=2}^{M-1}\binom{m}{2} = \binom{M}{3}.$$

Type IV: each of quadruple Whitehead products will have the following as a factor in its coefficient:

$$\sum_{m=3}^{M-1} \binom{m}{3} = \binom{M}{4}$$

In fact, by writing $(\alpha_k, \alpha_l)^M = (\alpha_k, \alpha_l)^{M-1}(\alpha_k, \alpha_l)$, a straightforward inductive argument shows that every iterated Whitehead product of weight m in $(\alpha_k, \alpha_l)^M$ has a coefficient with $\binom{M}{m} = \binom{M-1}{m-1} + \binom{M-1}{m}$ as a factor.

Therefore, for an arbitrary element $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots) \in [\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{H}P^n)]$, the non-zero coordinates of $\bar{\alpha}^M$ also fall into these four types as described above.

An estimate of $\exp_p [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)]$ has been given in [13, Theorem 3.8]. However, the argument that the exponent is bounded above by that of $\mathbb{S}^4$ in the proof of [13, Theorem 3.8] has a gap.

To give (better) estimates for $\exp_p [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)]$ we recall that a homotopy associative multiplication μ on an H-space X has a deviation from homotopy commutativity given by the difference $\mu - \mu \circ T : X \times X \to X$, where $T : X \times X \to X \times X$ is the map swapping terms. This is null homotopic when restricted to the wedge $X \vee X$ so gives a map $X \wedge X \to X$.

Theorem 3.1. *Let p be a prime and $r \geq 1$.*

(1) If $p \geq 5$ then $\exp_p [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq \exp_p \mathbb{H}P^\infty = \exp_p (\mathbb{S}^3) = p$;

(2) $\exp_3 [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq 9$;

(3) $\exp_2 [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq 8$.

Proof. (1): Type I elements are annihilated by $p = \exp_p \mathbb{H}P^\infty = \exp_p (\mathbb{S}^3)$.

There is an H-equivalence $\Omega(\mathbb{H}P^\infty) \simeq \mathbb{S}^3$, so with this multiplication the sphere $\mathbb{S}^3$ is homotopy associative. In this case, the deviation from commutativity for the homotopy associative multiplication on $\mathbb{S}^3$ is an element of $\pi_6(\mathbb{S}^3) \approx \mathbb{Z}_{12}$ which is zero when localized at $p \geq 5$. Thus, the multiplication is homotopy commutative so all Type II and Type III terms are trivial and we conclude that

$$\exp_p [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq \exp_p \mathbb{H}P^\infty \leq \exp_p (\mathbb{S}^3) = p$$

for $p \geq 5$ and $r \geq 1$.

(2): If $p = 3$ then, by [13, Proposition 3.7], there are no non-trivial elements of Type I or II as they are annihilated by $p = 3$. For Type III elements, $\binom{9}{3}$ is divisible by 3 so that $p^2 = 9$

annihilates elements of Type III. Thus,

$$\exp_3 [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq 9$$

for $r \geq 1$.

(3): If $p = 2$ then $\exp_2 \mathbb{H}P^\infty = \exp_2 \mathbb{S}^3 \leq 4$. Clearly, 2^3 annihilates elements of Type I. For Type II elements, $\binom{2^3}{2}$ is divisible by 4 so by [13, Proposition 3.7], there are no non-trivial Type II elements. Similarly, for Type III elements, $\binom{2^3}{3}$ is divisible by 4 so again by [13, Proposition 3.7], there are no non-trivial Type III elements. Hence,

$$\exp_2 [J(\mathbb{S}^r), \Omega(\mathbb{H}P^\infty)] \leq 8$$

for $r \geq 1$ and the proof is complete. □

Remark 3.2. It is known that there is a bijection between $[\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{S}^n)]$ and $\prod_{i\geq 1} \pi_{ir+1}(\mathbb{S}^n)$ as sets. When $r > 0$, certain homotopy groups of $\mathbb{S}^n$ do not appear in $\prod_{i\geq 1} \pi_{ir+1}(\mathbb{S}^n)$. It follows that the equalities obtained in [13, Theorem 2.1] should be replaced by inequalities in general while equalities hold when $r = 1$ (except for $p = 2$ and n even). For completeness, we state the correct version of [13, Theorem 2.1] below.

Theorem 3.3. *Let p be a prime.*

(1) *If $p \neq 2$ then*

$$\exp_p([\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{S}^n)]) \leq \exp_p (\mathbb{S}^n).$$

(2) *If $p = 2$ then*

$$\exp_2([\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{S}^n)]) \begin{cases} \leq \exp_2 (\mathbb{S}^n), & \text{if } n \text{ is odd;} \\ \leq 2\exp_2 (\mathbb{S}^n), & \text{if } n \text{ is even.} \end{cases}$$

In [13, Theorem 3.9], it was shown that [1]

$$\exp_p[\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{S}^{4n+3})] \leq \exp_p[\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{H}P^n)] \leq \exp_p[\Omega(\mathbb{S}^{r+1}), \Omega(\mathbb{S}^{4n+3})]\cdot\exp_p[\Omega(\mathbb{S}^{r+1}), \mathbb{S}^3]$$

for $n, r \geq 1$ and any prime p. Using our earlier calculations from Section 1 on the Whitehead products in $\mathbb{H}P^n$, we can give a much sharper estimate for the exponent of $[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)]$, thereby improving [13, Theorem 3.9]. First, note that $\exp_p (X_{(p)}) = \exp_p (X)$ for any nilpotent connected space X.

[1]The occurrence of $\mathbb{S}^{4n+3}$ was inadvertently stated as $\mathbb{S}^{2n+1}$ in [13, Theorem 3.9].

Theorem 3.4. *Let $n > 1$, $r \geq 1$ and p be a prime.*

(1) *If p is odd then* $\exp_p[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] = \exp_p[J(\mathbb{S}^r), \Omega(\mathbb{S}^{4n+3})] \leq \exp_p(\mathbb{S}^{4n+3}) = p^{2n+1}$*;*

(2) *if $p = 2$ then* $\exp_2[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] \leq \exp_2(\mathbb{S}^{4n+3}) \leq 2^{3n+2}$.

Proof. From Remark 1.18(6), we have $24[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}] = 0$ for $\gamma_{n,\mathbb{H}} : \Sigma\mathbb{S}^{4n+2} \to \mathbb{H}P^n$ and $i_{n,\mathbb{H}} : \Sigma\mathbb{S}^3 \hookrightarrow \mathbb{H}P^n$.

(1): Although the result for $p = 3$ is the same as for $p > 3$, the proof of the statement is different.

(i): If $p > 3$ then the p-localized Whitehead product $[\gamma_{n,\mathbb{H}}, i_{n,\mathbb{H}}]_{(p)} = 0$. Consequently, by means of the proof of [8, Theorem 1.1], we deduce an H-homotopy equivalence

$$\Omega\mathbb{H}P^n_{(p)} \simeq \Omega\mathbb{S}^{4n+3}_{(p)} \times \mathbb{S}^3_{(p)}.$$

Hence, Theorem 3.3 leads to

$$\exp_p[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] = \exp_p[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n_{(p)})] = \exp_p[J(\mathbb{S}^r), \Omega\mathbb{S}^{4n+3}_{(p)} \times \mathbb{S}^3_{(p)}] =$$

$$\max\{\exp_p[J(\mathbb{S}^r), \Omega(\mathbb{S}^{4n+3}_{(p)})], \exp_p[J(\mathbb{S}^r), \mathbb{S}^3_{(p)}]\} \leq \exp_p(\mathbb{S}^{4n+3}) = p^{2n+1}$$

for $n > 1$, $r \geq 1$.

(ii): If $p = 3$ then in view of Proposition 1.19, all iterated Whitehead products in the space $\mathbb{H}P^n_{(3)}$ are annihilated by 3 and, by Corollary 1.21, there are no non-trivial iterated Whitehead products of weight ≥ 4 on $\mathbb{H}P^n_{(3)}$. So, that we can follow the proof of [13, Theorem 2.1] (corrected as Theorem 3.3). There are only three types of elements in $[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)]$.

Type I elements are annihilated by $M = \exp_3 \mathbb{H}P^n = \exp_3(\mathbb{S}^{4n+3}) = 3^{2n+1}$. It is straightforward to verify that M divides $\binom{M}{2}$ and 3 divides $\binom{M}{3}$ since $n \geq 1$. It follows that there are no non-trivial elements of Type II or Type III. Consequently, we conclude that

$$\exp_3[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] \leq \exp_3(\mathbb{S}^{4n+3}) = 3^{2n+1}$$

for $n > 1$, $r \geq 1$.

(2): If $p = 2$ then, in view of Proposition 1.20, all Whitehead products in the space $\mathbb{H}P^n_{(2)}$ are annihilated by 2 provided n is odd. All double Whitehead products in the space $\mathbb{H}P^n_{(2)}$ are annihilated by 8 and all triple and quadruple Whitehead products by 2 provided n is even. It follows from the work of James and of Mahowald [22] that

$$\exp_2(\mathbb{S}^{2k+1}) \geq \lambda \cdot \exp_2(\mathbb{S}^{2k-1}),$$

where

$$\lambda = \begin{cases} 4, & \text{if } k \equiv 1 \pmod 4; \\ 2, & \text{if } k \equiv 0 \text{ or } 2 \pmod 4; \\ 1, & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Since $\mathbb{S}^{4n+3} = \mathbb{S}^{2(2n+1)+1}$ and $n > 1$, it follows that $\exp_2(\mathbb{S}^{2k+1}) = 2^t$ for some $t > 5$. Let $M = \exp_2(\mathbb{S}^{4n+3})$. Next, we consider two cases.

(i): If n is odd then, by Proposition 1.20, there are no non-trivial quadruple Whitehead products in $\mathbb{H}P^n_{(2)}$. We analyze elements of Type I, II and III. Note that M annihilates Type I elements. Since $M = 2^t$ for some $t > 5$, $\binom{M}{2}$ is divisible by 8 so there are no non-trivial Type II elements. For $M = 2^t$, $\binom{M}{3}$ is divisible by 2 and thus there are no non-trivial Type III elements. Then, we are in the same case as in the cases when $p = 3$. Thus, in view of [22], we have

$$\exp_2[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] \leq \exp_2[J(\mathbb{S}^r), \Omega(\mathbb{S}^{4n+3})] \leq \exp_2(\mathbb{S}^{4n+3}) \leq 2^{3n+2}$$

for $n > 1$, $r \geq 1$.

(ii): If n is even then by Proposition 1.20, there are no non-trivial quintuple Whitehead products in $\mathbb{H}P^n_{(2)}$. Then, we analyze elements of Type I, II, III and IV.

Again, $M = \exp_2(\mathbb{S}^{4n+3}) = 2^t$. Similar to the odd n case, there are no non-trivial elements of Type I, II, and III. For quadruple Whitehead products, since $t > 5$, 2 divides $\binom{M}{4}$ so by Prop. 1.20 there are no non-trivial elements of Type IV. Thus, in view of [22], we have

$$\exp_2[J(\mathbb{S}^r), \Omega(\mathbb{H}P^n)] \leq \exp_2[J(\mathbb{S}^r), \Omega(\mathbb{S}^{4n+3})] \leq \exp_2(\mathbb{S}^{4n+3}) \leq 2^{3n+2}$$

for $n > 1, r \geq 1$ and the proof is complete. □

Remark 3.5. Note that $\mathbb{H}P^1 = \mathbb{S}^4$. Thus, the exponents $\exp_p[J(\mathbb{S}^r), \Omega(\mathbb{H}P^1)]$ are given by Theorem 3.3 for $\mathbb{S}^4$.

4. Iterated Samelson product in symplectic groups Sp(m)

Given a topological group G, consider the Samelson product

$$\langle -, - \rangle : \pi_k(G) \times \pi_l(G) \longrightarrow \pi_{k+l}(G)$$

defined by the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^{k+l} = \mathbb{S}^k \wedge \mathbb{S}^l & \xrightarrow{\alpha\wedge\beta} & G \wedge G \\ & \searrow^{\langle -,- \rangle} & \big\downarrow {\scriptstyle [-,-]} \\ & & G \end{array}$$

where $[-,-]$ is the commutator map.

With each $\lambda \in \pi_1(G)$, we associate the operator

$$D_\lambda : \pi_k(G) \to \pi_{k+1}(G)$$

defined by taking the Samelson product with λ.

There is some evidence to support the following

Conjecture 4.1. (I.M. James [18]) For some values of s, depending on λ but not on k, the operator

$$D^s_\lambda : \pi_k(G) \to \pi_{k+s}(G)$$

defined by the s-th iteration of D_λ, is trivial.

James [18] proved that the conjecture is true for the rotation group R_t and the generator of $\pi_1(R_t) \approx \mathbb{Z}_2$ for $t \geq 3$.

Let $\mathrm{Sp}(n)$ be the n-th symplectic group, write $X_{n,k} = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ for the quaternionic Stiefel manifold, $p : \mathrm{Sp}(n) \to X_{n,k}$ and $i : \mathrm{Sp}(n) \hookrightarrow \mathrm{Sp}(n+k)$ for the natural projection and the inclusion maps, respectively.

$$\begin{array}{ccccc} & & \pi_s(\mathrm{Sp}(n)) \times \pi_t(\mathrm{Sp}(m)) & & \\ & \swarrow^{p_* \times p_*} & & \searrow^{i* \times i_*} & \\ \pi_s(X_{n,k}) \times \pi_t(X_{m,k}) & & & & \pi_s(\mathrm{Sp}(n+m-k)) \times \pi_t(\mathrm{Sp}(n+m-k)) \\ \big\downarrow {\scriptstyle \star} & & & & \big\downarrow {\scriptstyle \langle -,- \rangle} \\ \pi_{s+t+1}(X_{n+m,k}) & & \xrightarrow[\partial]{} & & \pi_{s+t}(\mathrm{Sp}(n+m-k)) \end{array}$$

Recall the relation given Bott [5] and Husseini [16]:

$$\big\langle i_*\alpha, i_*\beta\big\rangle = \pm\partial((p_*\alpha)\star(p_*\beta)), \tag{4.2}$$

where $\star$ is the intrinsic join defined by James [17], $\alpha \in \pi_s(\mathrm{Sp}(m))$, $\beta \in \pi_t(\mathrm{Sp}(n))$ and ∂ is an appropriate connecting map.

Let now $i : \mathrm{Sp}(1) = \mathbb{S}^3 \hookrightarrow \mathrm{Sp}(m)$ be the canonical inclusion. Given $\alpha' \in \pi_t(\mathrm{Sp}(1))$, consider

$$D_\alpha : \pi_k(\mathrm{Sp}(m)) \longrightarrow \pi_{k+t}(\mathrm{Sp}(m))$$

defined by taking the Samelson product with $\alpha = i_*\alpha'$. Kachi [19, Theorem 3.2] has proved

Theorem 4.3. *If $\pi_3(\mathrm{Sp}(1)) = \mathbb{Z}\{\alpha'\}$ and $\alpha = i_*\alpha'$ then the iterated operator*

$$D^s_\alpha : \pi_k(\mathrm{Sp}(m)) \longrightarrow \pi_{k+3s}(\mathrm{Sp}(m))$$

of D_α is trivial for any k, where

(1) *$s = 2$ if $m \equiv -1 \pmod{24}$;*

(2) *$s = 3$ if $m = 1$ and $s = 2$ if $m \geq 3$ is odd;*

(3) *$s = 5$ if m is even.*

To generalize Theorem 4.3, we first extend [19, Lemma 2.1] as follows.

Lemma 4.4. *If $\alpha' \in \pi_t(\mathrm{Sp}(1))$ and $\alpha = i_*\alpha'$ then*

$$D_\alpha(\beta) = \pm\partial(\Sigma^4\beta \circ \Sigma^{k+1}(\alpha'))$$

for $\beta \in \pi_k(\mathrm{Sp}(m))$.

Proof. Let $\alpha' \in \pi_t(\mathrm{Sp}(1))$, $\alpha = i_*\alpha'$ and $\beta \in \pi_k(\mathrm{Sp}(m))$. First, notice that the intrinsic join $(p_*\alpha)\star(p_*\beta) = \alpha\star(p_*\beta) = \Sigma(\alpha' \wedge p_*(\beta)) = \Sigma(\alpha') \wedge p_*(\beta)$. Then, by [25, Proposition 3.1], we have $(p_*\alpha)\star(p_*\beta) = \pm\Sigma^4 p_*\beta \circ \Sigma^{k+1}(\alpha')$. Hence, (4.2) leads to

$$D_\alpha(\beta) = \pm\partial(\Sigma^4 p_*\beta \circ \Sigma^{k+1}(\alpha'))$$

for $\beta \in \pi_k(\mathrm{Sp}(m))$ and the proof is complete. □

Now, we are in a position to state

Theorem 4.5. *If $\alpha_l' \in \pi_{t_l}(\mathrm{Sp}(1))$ and $\alpha_l = i_*\alpha_l'$ for $k = 1, \ldots, s$ then the iterated operator*

$$D_{\alpha_s} \circ \cdots \circ D_{\alpha_1} : \pi_k(\mathrm{Sp}(m)) \longrightarrow \pi_{k+t_1+\cdots+t_s}(\mathrm{Sp}(m))$$

is trivial for any k, where

(1) $s = 2$ *if* $m \equiv -1 \pmod{24}$;

(2) $s = 3$ *if m is odd;*

(3) $s = 5$ *if m is even.*

Proof. Recall that $p_*\partial = \Delta$ is the boundary map in the exact sequence of the fibration

$$\mathbb{S}^{4m-1} \to \mathrm{Sp}(m+1)/\mathrm{Sp}(m-1) \to \mathbb{S}^{4m+3}.$$

Next, we need the well known formula

$$\Delta(\delta \circ \Sigma(\gamma)) = \Delta(\delta) \circ \gamma, \tag{4.6}$$

By the proof of [19, Theorem 3.2], we have

$$\Delta(\iota_{4m+3}) = \begin{cases} \omega \text{ for } m = 1; \\ (m+1)\nu_{4m-1} \text{ for } m \geq 2. \end{cases} \tag{4.7}$$

Now, we make use of (4.2), Lemma 4.4 and (4.6) to derive that

$D_{\alpha_2} \circ D_{\alpha_1}(\beta) = \pm\partial(\Sigma^4(p_*(\partial(\Sigma^4(p_*(\beta)) \circ \Sigma^{k+1}(\alpha_1'))) \circ \Sigma^{t_1+k+1}(\alpha_2'))) = \pm\partial(\Sigma^4(\Delta(\Sigma^4(p_*(\beta))) \circ \Sigma^{k+1}(\alpha_1') \circ \Sigma^{t_1+k+1}(\alpha_2')) =$

$$\begin{cases} \pm\partial(\Sigma^4(\omega) \circ \Sigma^7(p_*(\beta)) \circ \Sigma^{k+4}(\alpha_1') \circ \Sigma^{t_1+k+1}(\alpha_2')) \text{ for } m = 1; \\ \pm\partial((m+1)\nu_{4m+3} \circ \Sigma^7(p_*(\beta)) \circ \Sigma^{k+4}(\alpha_1') \circ \Sigma^{t_1+k+1}(\alpha_2')) \text{ for } m \geq 2. \end{cases}$$

Thus,

$$D_{\alpha_2} \circ D_{\alpha_1} = 0 \text{ if } m \equiv -1 \pmod{24}.$$

Iterating once more, we have $D_{\alpha_3} \circ D_{\alpha_2} \circ D_{\alpha_1}(\beta) = \pm\partial((m+1)^2\nu_{4m+3}^2 \circ \Sigma^{10}(p_*(\beta)) \circ \Sigma^{k+7}(\alpha_1') \circ \Sigma^{t_1+k+4}(\alpha_2') \circ \Sigma^{t_1+t_2+k+1}(\alpha_3'))$. Since $2\nu_{4m+3}^2 = 0$, it follows that

$$D_{\alpha_3} \circ D_{\alpha_2} \circ D_{\alpha_1} = 0$$

for m odd.

After two more iterations, we have

$$D_{\alpha_5} \circ D_{\alpha_4} \circ D_{\alpha_3} \circ D_{\alpha_2} \circ D_{\alpha_1} = 0$$

since $\nu^4_{4m+3} = 0$ and the proof is complete. $\square$

Theorem 4.5 leads to the following generalization of [1, Lemma 5] (which is really a consequence of [21]):

Corollary 4.8. *If $\alpha_l \in \pi_{t_k}(\mathbb{S}^3)$ for $l = 1, 2, 3, 4$ then the Samelson product*

$$\big\langle \alpha_4, \big\langle \alpha_3, \big\langle \alpha_2, \alpha_1 \big\rangle \big\rangle \big\rangle = 0 \in \pi_{t_1+t_2+t_3+t_4}(\mathbb{S}^3).$$

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